# TRANSMISSION BOUNDARY VALUE PROBLEMS IN ELASTICITY

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**Abstract**

We address the issue of sharpness of the well-posedness results for $L^p$ transmission boundary value problems in non-smooth domains, for the Laplacian and the system of elastostatics. Our approach relies on Mellin transform techniques for singular integrals naturally associated with the transmission problems and on a careful analysis of the $L^p$ spectra of such singular integrals.

## 1 Introduction

Two classical boundary value problems for the Laplace operator in a given domain $\Omega \subset \mathbb{R}^n$ are the Dirichlet and Neumann problems

$$(D) \quad \begin{cases} \Delta u = 0, & \text{in} \quad \Omega, \\ u\Big|_{\partial\Omega} = f & \text{on} \quad \partial\Omega, \end{cases} \tag{1}$$

$$(N) \quad \begin{cases} \Delta u = 0, & \text{in} \quad \Omega, \\ \partial_\nu u = f & \text{on} \quad \partial\Omega, \end{cases} \tag{2}$$

where $\nu$ is the outward unit normal vector to $\partial\Omega$ and $\partial_\nu$ denotes the derivative in the normal direction. In [2], B. Dahlberg shows that in the case when $\Omega$ is of class $\mathcal{C}^1$, the problem $(D)$ in (1) is well posed for all $p \in (1, \infty)$. The approach taken relied on a careful analysis of the properties of the harmonic measure on $\mathcal{C}^1$ domains and the theory of weighted norm inequalities. When, however, the domain $\Omega$ is merely Lipschitz, B. Dahlberg established that the well-posedness range for $(D)$ is $p \in (2 - \varepsilon, \infty)$, where $\varepsilon$ depends on $\Omega$. Relatively simple counterexamples in [8] show that the well-posedness range $p \in [2, \infty)$ is sharp in the class of Lipschitz domains. Dahlberg's method relied on positivity and the Harnack principle, and could not be applied for the treatment of the Neumann problem $(N)$ nor for Dirichlet problems for systems of equations.

A unified approach for the treatment of $(D)$ and $(N)$ is the method of layer potentials. By employing the classical Fredholm theory when $\Omega$ is of class $\mathcal{C}^{1+\varepsilon}$, one has that both problems are well-posed for all $p \in (1, \infty)$. Whether the Fredholm theory applies also to the case of $\mathcal{C}^1$ domains is a far more delicate issue and it has been completely resolved by E. Fabes, M. Jodeit and N. Riviere in [7]. This yields $p \in (1, \infty)$ as the well-posedness range for (1) and (2) with data in $L^p(\partial\Omega)$ in the class of $\mathcal{C}^1$ domains. Turning attention to the class of Lipschitz domains, B. Dahlberg and C. Kenig in [3] prove that $p \in (1, 2]$ is the well-posedness range of the Neumann problem $(N)$. Again, as shown in [8], this range is optimal.

Another important type of boundary conditions for the Laplacian, arising naturally in inverse scattering, is that of transmission conditions. The associate boundary value problem reads

$$(TBVP) \quad \begin{cases} \Delta u_\pm = 0 \quad \text{in} \quad \Omega^\pm \\ M(\nabla u_\pm) \in L^p(\partial\Omega) \\ u_+\big|_{\partial\Omega} - u_-\big|_{\partial\Omega} = 0 \\ \partial_\nu u_+ - \gamma \partial_\nu u_- = g \in L^p(\partial\Omega), \end{cases} \tag{3}$$

where $\gamma \in (0, 1)$ is the transmission coefficient, $M$ is the nontangential maximal operator, $\Omega_+ := \Omega$ and $\Omega_- := \mathbb{R}^n \setminus \bar{\Omega}$ (with $\bar{\Omega}$ denoting the closure in $\mathbb{R}^n$). Furthermore, $\partial_\nu$ stands for the normal derivative on $\partial\Omega$. Recently, in [5], the authors show that, for any $\gamma \in (0, 1)$, the problem $(TBVP)$ is well-posed (uniqueness understood modulo constants) in the class of (special) Lipschitz domains for every $p \in (1, 2]$. In this note we answer the question, posed to us by L. Escauriaza and M. Mitrea, whether this range is optimal in the class of domains under consideration. Indeed, we prove that

**Theorem 1.1.** *For any $p > 2$ there exist a special Lipschitz domain $\Omega$ and $\gamma \in (0,1)$ such that $(TBVP)$ is not well-posed.*

Our counterexamples, constructed in the simplest geometric context, i.e., when $\Omega$ is an infinite sector in $\mathbb{R}^2$ of a sufficiently small aperture $\theta \in (0, 2\pi)$, rely on a careful analysis of the $L^p$ spectra of an integral operator which is naturally associated with $(TBVP)$.

## 2 Preliminaries

**Definition 2.1.** *A domain $\Omega \subset \mathbb{R}^2$ lying above the graph of a Lipschitz function $\phi : \mathbb{R} \to \mathbb{R}$ is called a special Lipschitz domain. That is*

$$\Omega := \{X = (x_1, x_2) \in \mathbb{R}^2 : \ x_2 > \phi(x_1)\}. \tag{4}$$

For the remainder of the paper we denote by $d\sigma$ the surface measure on $\partial\Omega$, and by $\nu$ the outward unit normal vector which exists almost everywhere with respect to $d\sigma$. As before, we set $\Omega_+ := \Omega$ and $\Omega_- := \mathbb{R}^2 \setminus \bar{\Omega}$. Next, for any $P \in \partial\Omega$, we introduce the non-tangential approach regions with vertex at $P$ as

$$\Upsilon^{\pm}(P) := \{X \in \Omega_{\pm} : |P - X| < \kappa \mathrm{dist}(X, \partial\Omega)\}, \tag{5}$$

where $\kappa > 1$ is a fixed, sufficiently large constant. The cone-like regions defined in (5) are then used to define non-tangential traces on $\partial\Omega$. Specifically, if $u_{\pm} : \Omega_{\pm} \to \mathbb{R}$ we let, for a.e. $P \in \partial\Omega$,

$$u_{\pm}\big|_{\partial\Omega}(P) := \lim_{\substack{X \in \Upsilon^{\pm}(P) \\ X \to P}} u_{\pm}(X), \tag{6}$$

and, again for a.e. $P \in \partial\Omega$,

$$\partial_\nu u_{\pm}(P) := \langle \nu(P), (\nabla u_{\pm})\big|_{\partial\Omega}(P) \rangle. \tag{7}$$

Here and elsewhere $\langle \cdot, \cdot \rangle$ stands for the canonical inner product in $\mathbb{R}^2$. Next, we recall the non-tangential maximal operator $M$ acting on functions $u_{\pm} : \Omega_{\pm} \to \mathbb{R}$ which is given at each boundary point $P \in \partial\Omega$ by

$$M(u_{\pm})(P) := \sup\{|u_{\pm}(X)| : \ X \in \Upsilon^{\pm}(P)\}. \tag{8}$$

For each $1 < p < \infty$, the space $L^p(\partial\Omega)$ is the Lebesgue space of $p$-integrable functions on $\partial\Omega$ with respect to the surface measure $d\sigma$, i.e.

$$L^p(\partial\Omega) = \left\{ f : \int_{\partial\Omega} |f|^p d\sigma < +\infty \right\}. \tag{9}$$

Also , let

$$L^p_1(\partial\Omega) := \{f \in L^p(\partial\Omega) : \partial_\tau f \in L^p(\partial\Omega)\},$$
$$\dot{L}^p_1(\partial\Omega) := \{f \in L^p_{\mathrm{loc}}(\partial\Omega) : \partial_\tau f \in L^p(\partial\Omega)\}/\mathbb{R},$$

where $\partial_\tau$ is the tangential derivative along $\partial\Omega$ and $L^p_{\mathrm{loc}}(\partial\Omega)$ is the local version of the space $L^p(\partial\Omega)$. If $[g] \in \dot{L}^p_1(\partial\Omega)$ denotes the equivalence class of $g$, we set

$$\|[g]\|_{\dot{L}^p_1(\partial\Omega)} := \|\partial_\tau g\|_{L^p(\partial\Omega)}. \tag{10}$$

Now recall the definitions of the classical harmonic layer potential operators for a Lipschitz domain $\Omega \subset \mathbb{R}^2$. We start with the definition of $\mathcal{S}$, the single layer potential operator associated to the Laplacian, and its boundary version $S$. Specifically, fix $X_0 \notin \partial\Omega$ and for $f : \partial\Omega \longrightarrow \mathbb{R}$ set

$$\mathcal{S}f(X) := \frac{1}{2\pi} \int_{\partial\Omega} [\log|X - Y| - \log|X_0 - Y|] f(Y)\, d\sigma(Y),$$

where $X \in \mathbb{R}^2 \setminus \partial\Omega$, and

$$Sf(X) := \frac{1}{2\pi} \int_{\partial\Omega} [\log|X - Y| - \log|X_0 - Y|] f(Y)\, d\sigma(Y),$$

where $X \in \partial\Omega$. We shall also work with $K^*$, the formal adjoint of the boundary version of the double layer potential operator, given by

$$K^* f(P) := p.v. \frac{1}{\pi} \int_{\partial\Omega} \frac{\langle P - Y, \nu(P) \rangle}{|P - Y|^2} f(Y)\, d\sigma(Y), \tag{11}$$

where $P \in \partial\Omega$ and *p.v.* indicates that the integral is taken in the principal value sense. The following result which is going to be useful for us in the sequel (cf., [1], [15]).

**Theorem 2.2.** *Let $\Omega \subset \mathbb{R}^2$ be a special Lipschitz domain and assume that $1 < p < \infty$.*

*(1) The single layer potential operator $\mathcal{S}$ satisfies*

$$\mathcal{S}f|_{\partial\Omega_+} = \mathcal{S}f|_{\partial\Omega_-} = Sf \quad \text{and} \quad \|\mathcal{N}(\nabla \mathcal{S}f)\|_{L^p(\partial\Omega)} \le C\|f\|_{L^p(\partial\Omega)}, \tag{12}$$

*uniformly for $f \in L^p(\partial\Omega)$.*

*(2) Given $f \in L^p(\partial\Omega)$, for almost every $P \in \partial\Omega$ we have*

$$\langle \nu(P), \lim_{\substack{X\in\Upsilon^{\pm}(P)\\ X\to P}} \nabla \mathcal{S}f(X)\rangle = (\mp\tfrac{1}{2}I + K^*)f(P). \quad (13)$$

*(3) The operator $K^*$ acting from $L^p(\partial\Omega)$ into $L^p(\partial\Omega)$ is bounded.*

Finally, if $\mathcal{X}$ is a Banach space and $T : \mathcal{X} \to \mathcal{X}$ is a linear, bounded operator, we denote by $\sigma(T;\mathcal{X})$ the spectrum of the operator $T$ given by

$$\sigma(T;\mathcal{X}) := \{z \in \mathbb{C} \, : \, zI - T \text{ not invertible on } \mathcal{X}\},$$

where $I$ denotes the identity.

## 3 Spectral properties of layer potentials

The goal of this section is to present a more general version of Theorem 1.1 adapted to the case of transmission boundary value problems for the Lamé system of elastostatics. In general, consider a second order elliptic system in $\mathbb{R}^n$ with constant coefficients given by

$$(\mathcal{L}\vec{u})^\alpha := a_{ij}^{\alpha\beta}\partial_i\partial_j u^\beta, \qquad a_{ij}^{\alpha\beta} \in \mathbb{R}, \quad (14)$$

where

$$\begin{aligned} &\vec{u} = (u^1, \cdots, u^m), \\ &i,j = 1,\cdots,n, \qquad \alpha,\beta = 1,\cdots,m. \end{aligned} \quad (15)$$

In (15) we use the standard Einstein convention for summation over repeated indices. The collection of coefficients

$$\mathcal{A} := (a_{ij}^{\alpha\beta})_{\alpha,\beta,i,j} \quad (16)$$

is referred to as the coefficient tensor of the system (14) and is assumed to satisfy the symmetry condition

$$a_{ij}^{\alpha\beta} = a_{ji}^{\beta\alpha}, \quad (17)$$

for any $i,j = 1,\cdots,n$, $\alpha,\beta = 1,\cdots,m$, and the ellipticity condition, also known as the Legendre-Hadamard property. The latter asserts that there exists $c > 0$ such that for any $\xi = (\xi_i) \in \mathbb{R}^n$ and $\eta = (\eta^\alpha) \in \mathbb{R}^m$, the following holds

$$a_{ij}^{\alpha\beta}\xi_i\xi_j\eta^\alpha\eta^\beta \geq c|\xi|^2|\eta|^2. \quad (18)$$

The conormal derivative of a vector valued function $\vec{u}$ depends on the choice of coefficient tensor and is given by

$$(\partial_{\nu_\mathcal{A}}\vec{u})^\alpha := \nu_i a_{ij}^{\alpha\beta}\partial_j u^\beta. \quad (19)$$

Thus, the transmission boundary value problem associated to $\mathcal{L}$ and the coefficient tensor $\mathcal{A}$ reads

$$(TBVP) \quad \begin{cases} \mathcal{L}\vec{u}_\pm = \vec{0} \quad \text{in} \quad \Omega^\pm \\ M(\nabla\vec{u}_\pm) \in (L^p(\partial\Omega))^m \\ \vec{u}_+\big|_{\partial\Omega} - \vec{u}_-\big|_{\partial\Omega} = \vec{0} \\ \partial_{\nu_\mathcal{A}}\vec{u}_+ - \gamma\partial_{\nu_\mathcal{A}}\vec{u}_- = \vec{g} \in (L^p(\partial\Omega))^m, \end{cases}$$

where $\nu$ is the outward normal unit vector and $\gamma \in (0,1)$ is the transmission coefficient.

Consider next the Lamé system of elastostatics in two dimensions. The operator $\mathcal{L}\vec{u}$ has the form

$$\mathcal{L}\vec{u} = \mu\triangle\vec{u} + (\lambda+\mu)\nabla\text{div}\vec{u} = 0, \quad (20)$$

where $\lambda,\mu$ are the so-called Lamé moduli and satisfy

$$\mu > 0 \quad \text{and} \quad \lambda + \mu \geq 0. \quad (21)$$

Note that $\mathcal{L}$ admits the writing (14) for any member of the infinite family of coefficient tensors $a_{ij}^{\alpha\beta}$ of the type

$$\mathcal{A}(r) = (a_{ij}^{\alpha\beta}(r))_{\alpha,\beta,i,j}, \quad (22)$$

where

$$a_{ij}^{\alpha\beta}(r) = \mu\delta_{ij}\delta\alpha\beta + (\mu+\lambda-r)\delta_{i\alpha}\delta_{j\beta} + r\delta_{i\beta}\delta_{j\alpha} \quad (23)$$

for any $r \in \mathbb{R}$. We point out that for each $r \in \mathbb{R}$, $\mathcal{A}(r)$ satisfies the symmetry property (17) and the Legendre-Hadamard condition (18).

Two particular choices for the parameter $r$ in (22) arise naturally in physical applications. The first,

$$r_1 := \frac{\mu(\lambda+\mu)}{(3\mu+\lambda)}, \quad (24)$$

gives rise to the pseudo-stress conormal derivative, and the second,

$$r_2 := \mu, \quad (25)$$

leads to the traction conormal derivative. The corresponding transmission boundary value problems are

$$(TBVP)_{\text{Lamé}} \quad \begin{cases} \mathcal{L}\vec{u}_{\pm} = \vec{0} \quad \text{in} \quad \Omega^{\pm} \\ M(\nabla \vec{u}_{\pm}) \in (L^p(\partial\Omega))^m \\ \vec{u}_+\big|_{\partial\Omega} - \vec{u}_-\big|_{\partial\Omega} = \vec{0} \\ \partial_{\nu_{\mathcal{A}(r_1)}}\vec{u}_+ - \gamma\partial_{\nu_{\mathcal{A}(r_1)}}\vec{u}_- = \vec{g}, \end{cases} \tag{26}$$

and

$$(TBVP)_{\text{Lamé}} \quad \begin{cases} \mathcal{L}\vec{u}_{\pm} = \vec{0} \quad \text{in} \quad \Omega^{\pm} \\ M(\nabla \vec{u}_{\pm}) \in (L^p(\partial\Omega))^m \\ \vec{u}_+\big|_{\partial\Omega} - \vec{u}_-\big|_{\partial\Omega} = \vec{0} \\ \partial_{\nu_{\mathcal{A}(r_2)}}\vec{u}_+ - \gamma\partial_{\nu_{\mathcal{A}(r_2)}}\vec{u}_- = \vec{g}. \end{cases} \tag{27}$$

In (26) and (27), $\vec{g}$ is an $m$-dimensional vector valued function with components in $L^p(\partial\Omega)$.

The fundamental solution for the Lamé system in two dimensions is

$$\Gamma(X) := \rho\delta_{ij}\log|X|^2 - \tau\frac{X_iX_j}{|X|^2} \tag{28}$$

where $i,j = 1,2$ and $\delta_{ij}$ is the canonical Kronecker symbol, and

$$\rho = \frac{3\mu+\lambda}{4\pi\mu(2\mu+\lambda)} \quad \text{and} \quad \tau = \frac{\mu+\lambda}{4\pi\mu(2\mu+\lambda)}. \tag{29}$$

We have $\mathcal{L}\Gamma = 2\delta_0 I_{2\times 2}$, where $I_{2\times 2}$ denotes the $2\times 2$ identity matrix and $\delta_0$ is the delta Dirac distribution with unit mass at 0. Then the formal adjoint of the boundary version of the double layer potential operator associated to $\mathcal{A}(r)$ is given by

$$K^*_{\mathcal{A}(r)}\vec{f}(P) = \int_{\partial\Omega}\frac{\partial\Gamma}{\partial\nu_{A(r)}}(P-Y)\vec{f}(Y)d\sigma(Y), \tag{30}$$

where the conormal derivative of the fundamental solution is taken column by column. Note that for the choice $\mu+\lambda = 0$ in (24) the operator $K^*_{\mathcal{A}(r_1)}$ becomes

$$K^*_{\mathcal{A}(0)} = \begin{pmatrix} K^* & 0 \\ 0 & K^* \end{pmatrix} \tag{31}$$

with $K^*$ as in (11). Due to the fundamental result of R. Coifman, A. McIntosh and Y. Meyer in [1], for $\Omega$ a bounded Lipschitz domain in $\mathbb{R}^n$ and $p \in (1,\infty)$, we have that for each $r \in \mathbb{R}$,

$$K^*_{\mathcal{A}(r)} : (L^p(\partial\Omega))^2 \to (L^p(\partial\Omega))^2 \tag{32}$$

is a linear and bounded operator. For the rest of the exposition we denote by

$$K^*_{\psi} := K^*_{\mathcal{A}(r_1)} \tag{33}$$

and

$$K^*_{\text{traction}} := K^*_{\mathcal{A}(r_2)}. \tag{34}$$

We present next a couple of results regarding spectral properties of the operators $K^*_{\psi}$ and $K^*_{\text{traction}}$ on $L^p$ spaces on the boundary of infinite sectors. For a proof of these results, see [10], [11]. To this end, let $\Omega$ be the unbounded domain consisting of the interior of an angle of measure $\theta \in (0,2\pi)$. In order to make the subsequent presentation somewhat easier to follow, we need more notation. Let us introduce the following

$$\begin{aligned} A &:= (1-\kappa(r))z\sin\theta, \\ B &:= \sin(z(\pi-\theta)), \\ C &:= \cos(z(\pi-\theta)), \\ D &:= \sin(z\pi), \\ E &:= \cos(z\pi). \end{aligned} \tag{35}$$

where

$$\kappa(r) = \frac{\mu(3\mu+\lambda) - r(\mu+\lambda)}{2\mu(2\mu+\lambda)}. \tag{36}$$

**Theorem 3.1.** *Consider the equation*

$$\begin{aligned} &\big((wD+AC)^2 - B^2 + (AB)^2\big)\times \\ &\qquad \big((wD-AC)^2 - B^2 + (AB)^2\big) = 0. \end{aligned} \tag{37}$$

*Then, for each* $1 < q < \infty$, $\sigma(K^*_{\psi};(L^q(\partial\Omega))^2)$ *consists of* $\{0\}$ *and the set of all* $w \in \mathbb{C}$ *such that (37) holds for some* $z = \frac{1}{p} + iy, y \in \mathbb{R}$ *. In particular, if*

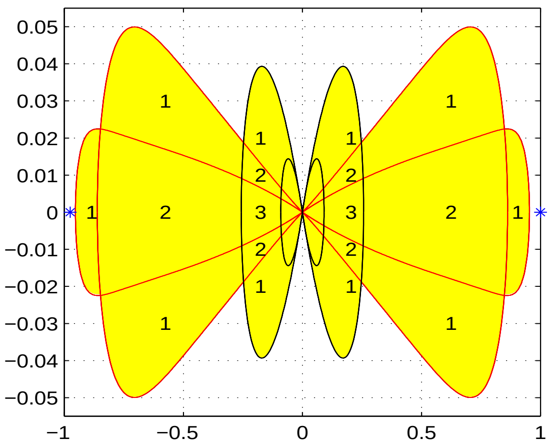
$$R_{\psi}(\theta,p,v) := \frac{\left|\frac{v}{p}\sin\theta\cos\frac{\omega}{p} + \sin\frac{\omega}{p}\sqrt{1-\frac{v^2}{p^2}\sin^2\theta}\right|}{2\sin\frac{\pi}{p}}, \tag{38}$$

*where* $\omega = \pi - \theta$, *then*

$$R_{\psi}(\theta,p,v) \in \sigma(K^*_{\psi};(L^q(\partial\Omega))^2). \tag{39}$$

*Here,* $1/p + 1/q = 1$ *and* $v = \frac{\mu+\lambda}{3\mu+\lambda} = 1-\kappa(r_1)$ *where* $r_1$ *is as in (24).*

Below is a figure the spectrum of $K_{\psi}^{*}$ on a sector for $p = 6$ and $\theta = \frac{\pi}{10}$ and $\frac{5\pi}{6}$.




Also,

**Theorem 3.2.** *The spectrum of the operator* $K^{*}_{\text{traction}}$ *on* $(L^q(\partial\Omega))^2$, $1 < q < \infty$, *is given by*

$$\sigma\left(K^{*}_{\text{traction}}; (L^q(\partial\Omega))^2\right) = \{w \in \mathbb{C}; (wD \pm AC)^2 = Q_{\pm}\} \cup \{-L, L\}, \tag{40}$$

*for some* $z \in \frac{1}{p} + iy$, $y \in \mathbb{R}$, *where* $1/p + 1/q = 1$, *and*

$$Q_{\pm} = B^2 + L^2C^2 - (LE \mp AB)^2 \tag{41}$$

*and*

$$L = \frac{\mu}{2\mu + \lambda}. \tag{42}$$

*In particular, if*

$$R_{\text{traction}}(\theta, L, p) = \left(\frac{1}{2\sin\frac{\pi}{p}}\right) \times \left\{\sqrt{E(\theta, p, L)} + \tfrac{1}{p}(1-L)\sin\theta\cos(\tfrac{\omega}{p})\right\}, \tag{43}$$

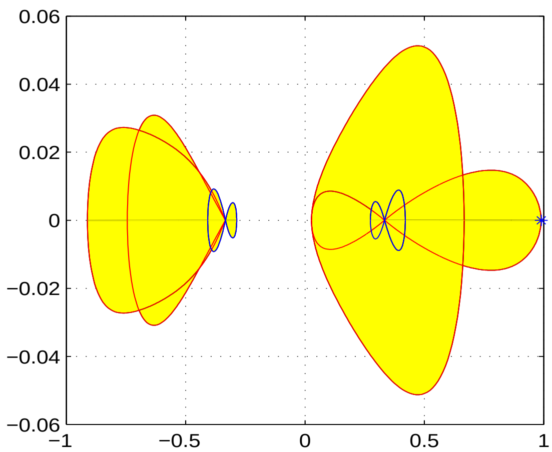
*where*

$$E(\theta, L, p) = \sin^2\left(\frac{\omega}{p}\right) + L^2\cos^2\left(\frac{\omega}{p}\right) - \left[L\cos\left(\frac{\pi}{p}\right) - \frac{1}{p}(1-L)\sin\theta\sin\left(\frac{\omega}{p}\right)\right]^2, \tag{44}$$

*and* $\omega = \pi - \theta$, *then*

$$R_{\text{traction}}(\theta, L, p) \in \sigma(K^{*}_{\text{traction}}; (L^q(\partial\Omega))^2). \tag{45}$$

The following figure gives an example of the $L^p$ spectrum of $K^{*}_{\text{traction}}$ on a sector in the case $p = 10$ and $\theta = \frac{\pi}{5}$ and $\theta = \frac{9\pi}{10}$.




## 4 The main result

In this section we sketch the proof of the main results of this note. We have

**Theorem 4.1.** *For every* $p > 2$ *and* $\lambda, \mu$ *as in (21) there exists a special Lipschitz domain* $\Omega$ *and* $\gamma \in (0, 1)$ *such that* $(TBVP)_{Lamé}$ *in (26) and (27) are not well-posed.*

The proof follows from the following sequence of technical lemmas.

First, fix $\Omega$ to be an infinite sector of aperture $\theta \in (0, 2\pi)$, $\lambda, \mu$ Lamé moduli satisfying (21), and $r \in \{r_1, r_2\}$. Hereafter we let $p_{critic}(\theta, \lambda, \mu, r) \in (0, \infty)$ be such that $I + K^{*}_{A(r)}$ fails to be invertible on $L^{p_{critic}}(\partial\Omega)$.

**Lemma 4.2.** *Let $\Omega \subset \mathbb{R}^2$ be an infinite sector of aperture $\theta \in (0, 2\pi)$ and $\lambda, \mu$ as in (21). Then for $r \in \{r_1, r_2\}$ and each $p \in (1, \infty) \setminus \{p_{critic(\theta,\lambda,\mu,r)}\}$ the following implication holds*

$$\left.\begin{array}{l} \mathcal{L}\vec{u} = \vec{0} \text{ in } \quad \Omega \\ M(\nabla\vec{u}) \in (L^p(\partial\Omega))^2 \end{array}\right\} \Longrightarrow \vec{u} = \mathcal{S}\vec{f} + \vec{c} \text{ in } \Omega, \tag{46}$$

*for some $\vec{f} \in (L^p(\partial\Omega))^2$ and $\vec{c} \in \mathbb{R}^2$.*

Before stating the next lemma let us introduce one more piece of notation. For each $\gamma \in (0, 1)$ we set

$$\beta(\gamma) := \frac{1}{2} \cdot \frac{1+\gamma}{(1-\gamma)}. \tag{47}$$

**Lemma 4.3.** *Let $\Omega \subset \mathbb{R}^2$ be an infinite sector of aperture $\theta \in (0, 2\pi)$, $\lambda, \mu$ as in (21), and $\gamma \in (0, 1)$. Then, for $r \in \{r_1, r_2\}$ and every $p \in (1, \infty) \setminus \{p_{critic}(\theta, \lambda, \mu, r)\}$ such that the problems (26) and (27) are well-posed, it follows that*

$$\beta(\gamma) \notin \sigma(K^*_{\mathcal{A}(r)}; (L^p(\partial\Omega))^2). \tag{48}$$

Finally, we have

**Lemma 4.4.** *For each $p > 2$, $\lambda, \mu$ satisfying (21), and $r \in \{r_1, r_2\}$ there exists $\theta \in (0, 2\pi)$ and $\gamma \in (0, 1)$ such that $p_{critic}(\theta, \lambda, \mu, r) \neq p$ and, if $\Omega \subset \mathbb{R}^2$ is an infinite sector of aperture $\theta$, then the operator $\beta(\gamma)I - K^*_{\mathcal{A}(r)}$ is not invertible on $(L^p(\partial\Omega))^2$.*

Proofs of the previous results are to appear in [12]. Furthermore, due to the reduction (31), the main result implies Theorem 1.1.


## Acknowledgements

Katharine Ott was supported in part by a Virginia Space Grant Consortium Aerospace Graduate Research Fellowship. Irina Mitrea was supported in part by NSF Grant DMS 0513173 and a University of Virginia FEST Grant.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