

BENDING OF STRAIN-STIFFENING RUBBER-LIKE BEAMS

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Abstract

This paper is concerned with investigation of the effects of strain-stiffening for the classical problem of plane strain bending by an end moment of a rectangular beam composed of an incompressible isotropic nonlinearly elastic material. For a variety of specific strain-energy densities that give rise to strain-stiffening in the stress-stretch response, the stresses and resultant moments are obtained explicitly. While such results are well known for classical constitutive models such as the Mooney-Rivlin and neo-Hookean models, our primary focus is on materials that undergo *severe* strain-stiffening in the stress-stretch response. In particular, we consider in detail two phenomenological constitutive models that reflect limiting chain extensibility at the molecular level and involve constraints on the deformation. The amount of bending that beams composed of such materials can sustain is limited by the constraint. Aerospace applications include bending of vehicle tires, rubber seals, deployable space structures, and vibration absorbers.

1. Introduction

This paper is concerned with investigation of the effects of strain-stiffening for the classical problem of plane strain bending by an end moment of a rectangular beam composed of an incompressible isotropic nonlinearly elastic material. This problem has been widely investigated within the theory of finite hyperelasticity largely motivated by applications to rubber and rubber-like advanced materials. It has also been recognized that bending problems are of considerable interest in the context of biomechanics of soft tissues (see, e.g., Taber (2004)). Our particular focus here is on investigation of the stress response for special classes of constitutive models that give rise to severe *strain-stiffening* in their stress-stretch curves at large strains. The constitutive models that we employ reflect *limiting chain extensibility* at the molecular level and thus are appropriate for modeling non-crystallizing elastomers and soft biological tissues.

In the next Section, we discuss some preliminaries from the theory of nonlinear hyperelasticity for isotropic incompressible solids. In particular, we describe some phenomenological constitutive models that exhibit strain-stiffening at large strains. The first class of models reflects limiting chain

extensibility at the molecular level and gives rise to *severe* strain-stiffening in the stress-stretch response. The second class exhibits a less abrupt strain-stiffening. In Section 3, we summarize results for the problem of plane strain bending of a rectangular beam by end moments. This problem was one of the classical problems solved by Rivlin (1949a, b) for general incompressible isotropic elastic solids. In Section 4, we provide explicit expressions for the stresses, resultant moment and resultant out of plane force for the classical Mooney-Rivlin and neo-Hookean models. Our main focus is on results for strain-stiffening models and these are described in Section 5. Explicit analytic results are given for two limiting chain extensibility models that exhibit severe strain-stiffening and also for the exponential model. These results are compared with one another and with those for the classical models in Section 6.

2. Preliminaries

In continuum mechanics, the mechanical properties of elastomeric materials are described in terms of a strain-energy density function W per unit undeformed volume. If the left Cauchy-Green tensor is denoted by $\mathbf{B} = \mathbf{F}\mathbf{F}^T$, where \mathbf{F} is the gradient of the deformation and $\lambda_1, \lambda_2, \lambda_3$ are the principal stretches, then, for an isotropic material, W is a function of the strain invariants

$$I_1 = \text{tr}\mathbf{B}, I_2 = \frac{1}{2} \left[(\text{tr}\mathbf{B})^2 - \text{tr}(\mathbf{B}^2) \right], I_3 = \det \mathbf{B}. \quad (1)$$

Rubber-like materials are often assumed to be *incompressible* provided that the hydrostatic stress does not become too large and so the admissible deformations must be isochoric, i.e., $\det \mathbf{F} = 1$ so that $I_3 = 1$. The response of an incompressible isotropic elastic material can be determined by applying the standard constitutive law (see, e.g., Ogden, 1984; Beatty, 1987; Holzapfel, 2000)

$$\mathbf{T} = -p\mathbf{1} + 2 \frac{\partial W}{\partial I_1} \mathbf{B} - 2 \frac{\partial W}{\partial I_2} \mathbf{B}^{-1}, \quad (2)$$

where p is a hydrostatic pressure term associated with the incompressibility constraint and \mathbf{T} denotes the Cauchy stress.

The classical strain-energy density for *incompressible* rubber is the Mooney-Rivlin strain-energy

$$W^{MR} = \frac{1}{2} \mu [\alpha(I_1 - 3) + (1 - \alpha)(I_2 - 3)], \quad (3)$$

where $\mu > 0$ is the constant shear modulus for infinitesimal deformations and $0 < \alpha \leq 1$ is a dimensionless constant. When $\alpha = 1$ in (3), one obtains the neo-Hookean strain-energy

$$W^{nH} = \frac{\mu}{2} (I_1 - 3), \quad (4)$$

which corresponds to a Gaussian statistical mechanics model, and is often referred to as the kinetic theory of rubber. The theoretical predictions based on (3) do not adequately describe experimental data for rubber especially at high values of strain. For example, the strain-energy (3) is not able to describe the characteristic *S*-shaped load versus stretch curve exhibited in simple tension experiments.

Some phenomenological models that have been shown to be particularly useful in modeling severe strain-stiffening phenomena are those reflecting a maximum achievable length of the polymeric molecular chains composing the material. (See Horgan and Saccomandi, 2005, 2006 for reviews). More recent papers are those of Beatty (2007, 2008) who uses the term “limited elastic” for such materials. For isotropic incompressible materials, these can be described by strain-energies of the form $W(I_1, I_2, I^*)$ where I^* is a limiting chain extensibility parameter. The function W is such that the stress components are unbounded as $f(I_1, I_2, I^*) \rightarrow 0$ for some function f and so one must impose the *constraint*

$$f(I_1, I_2, I^*) < 0 \quad (5)$$

on admissible deformations.

One such model is a *two-parameter* generalized neo-Hookean (i.e., $W = W(I_1)$) model due to Gent (1996), who proposed the strain-energy density

$$W^G = -\frac{\mu}{2} J_m \ln \left(1 - \frac{I_1 - 3}{J_m} \right), \quad I_1 < J_m + 3, \quad (6)$$

where μ is the shear modulus for infinitesimal deformations and J_m is the limiting chain extensibility parameter. On using (2), we find that the Cauchy stress associated with (6) is given by

$$\mathbf{T} = -p\mathbf{I} + \mu \frac{J_m}{J_m - (I_1 - 3)} \mathbf{B}, \quad (7)$$

so that the stress has a singularity as $I_1 \rightarrow J_m + 3$, reflecting the rapid strain stiffening observed in experiments. The constraint (5) for this model is thus taken to be (6)₂. For rubber, typical values for the dimensionless parameter J_m for simple extension range from 30-100 whereas for biological tissue, much smaller values of J_m are appropriate. For example, for

human arterial wall tissue, values on the order of 0.4 to 2.3 have been suggested by Horgan and Saccomandi (2003). On taking the limit as $J_m \rightarrow \infty$ in (6) we recover the neo-Hookean model (4). Other related three-parameter models with dependence on I_2 are discussed in Gent (1996, 1999), Pucci and Saccomandi (2002), and in Ogden et al. (2004).

An alternative *two-parameter* limiting chain extensibility model with $W(I_1, I_2, J)$ was proposed by Horgan and Saccomandi (2004) where

$$W^N = -\frac{\mu}{2} \frac{(J-1)^2}{J} \ln \left(\frac{J^3 - J^2 I_1 + J I_2 - 1}{(J-1)^3} \right),$$

$$J I_1 - I_2 < (J^3 - 1)/J, \quad J > 1, \quad (8)$$

or, on using the principal stretches of the deformation,

$$W^N = \frac{\mu(J-1)^2}{2J} \ln \left(\frac{(1-\lambda_1^2/J)(1-\lambda_2^2/J)(1-\lambda_3^2/J)}{(1-1/J)^3} \right), \quad \lambda_1 \lambda_2 \lambda_3 = 1. \quad (9)$$

In (8) and (9), μ is the shear modulus for infinitesimal deformations. Note that the definitions of W^N here differ from those in Horgan and Saccomandi (2004, 2005, 2006) and in Horgan and Schwartz (2005) by a factor of $(J-1)^2/J^2$. The limiting chain extensibility parameter J is the square of the maximum stretch allowed by the finite extensibility of the chains so that

$$\max(\lambda_1^2, \lambda_2^2, \lambda_3^2) < J. \quad (10)$$

Again, in the limit as $J \rightarrow \infty$ in (8) or (9), we recover the neo-Hookean model (4). It is important to point out the difference between the *constraint* (10) and the *constraint* $I_1 < J_m + 3$ arising in connection with the Gent model. As already pointed out in Horgan and Saccomandi (2002a, 2006), the limiting chain condition expressed in terms of the principal invariant is less physically accessible than (10). Furthermore, the absence of the dependence on the *second* invariant in the basic Gent model entails some physical limitations.

Thus the W^N model has advantages over the basic Gent model. Note that (9) belongs to the class of models for which $W(\lambda_1, \lambda_2, \lambda_3, J)$, with $\lambda_1 \lambda_2 \lambda_3 = 1$ because of incompressibility. For such models, the limiting chain extensibility constraint is given in terms of the principal stretches directly and this has some advantages from a physical point of view. This alternative approach to constitutive model development reflecting limiting chain extensibility has been discussed by Horgan and Saccomandi (2002a), Murphy (2006) and Horgan and Murphy (2007). The response of the W^N model in *homogeneous* deformations such as simple extension, simple shear and equibiaxial extension was examined in Horgan and Schwartz (2005) and Kanner and Horgan (2007).

There is an important connection between the Gent and W^N models that we shall make use of later. It can be readily verified that, for *all* deformations for which

$$I_1 = I_2, \quad (11)$$

we have

$$W^G = W^N \quad (12)$$

if J_m in the definition (6) is formally replaced by \tilde{J} where

$$\tilde{J} \equiv (J-1)^2/J. \quad (13)$$

Of course this does not imply any relation between *derivatives* of the respective strain-energies with respect to the invariants. For the bending problem of concern here, we will show that (11) holds and so the equivalence result just described will apply.

While our primary concern here is with limiting chain extensibility models such as the above that exhibit *severe* strain-stiffening, we note that there are numerous strain-hardening constitutive models that have been successfully employed to investigate the effects of a *less abrupt* strain-stiffening. A generalized neo-Hookean model of this type widely used in the biomechanics literature is the two-parameter strain-energy density

$$W^F = \frac{\mu}{2b} \{ \exp[b(I_1-3)] - 1 \}, \quad (14)$$

where the dimensionless constant $b > 0$. This *exponential* model was first proposed by Fung (1967). On taking the limit as $b \rightarrow 0$ in (14) we recover the neo-Hookean model (4).

We observe that, while the exponential model reflects strain hardening, it does *not* exhibit the *rapid strain stiffening* characteristic of the limiting chain extensibility models. This is an important difference between these models. Chagnon et al. (2004) suggest that both types of models are essentially equivalent but as was discussed in Horgan and Saccomandi (2005, 2006) there are several significant differences in their predictions.

3. Plane strain bending of a rectangular bar

The problem of plane strain bending (flexure) of an incompressible rectangular bar into a curved circular-sector was one of the classic problems considered by Rivlin (1949a,b) for a general incompressible isotropic hyperelastic material (see also presentations in the books of Ogden, 1984; Lai et al., 1993 and Taber, 2004). The solution obtained by Rivlin is controllable i.e. valid for *all* incompressible isotropic hyperelastic solids. For convenience of the reader, in this Section we briefly present a derivation of the

general expressions for the stresses, resultant bending moment and resultant out-of-plane force necessary to maintain plane strain. Our approach is based on that of Lai et al. (1993).

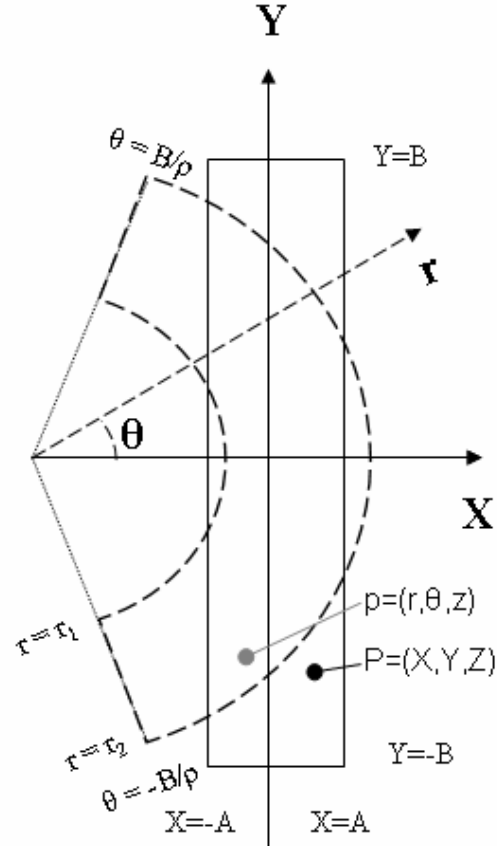


Fig. 1. Plane strain bending of a rectangular bar.

For the problem at hand (see Fig. 1), we consider the deformation field

$$r = f(X), \quad \theta = Y/\rho, \quad z = Z, \quad (15)$$

where $1/\rho$ is a parameter that controls the amount of deformation, and is closely related to the curvature of the deformed bar as will be shown later. The Cartesian coordinates (X, Y, Z) refer to the undeformed configuration with $-A \leq X \leq A$, $-B \leq Y \leq B$, and $-C \leq Z \leq C$, while the cylindrical coordinates (r, θ, z) denote points in the deformed configuration. Note from Fig.1 that the origin of the Cartesian coordinate system lies at the center of the undeformed bar, while the origin of the cylindrical coordinate system lies outside the bar.

On using (15), we find that the components of the deformation gradient tensor \mathbf{F} are given by

$$\mathbf{F} = \begin{pmatrix} df/dX & 0 & 0 \\ 0 & f/\rho & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (16)$$

For an incompressible material we require $\det \mathbf{F} = 1$ and so we obtain the differential equation

$$(df/dX)(f/\rho) = 1, \quad (17)$$

which may easily be solved for $f(X)$ to yield

$$r = \sqrt{2\rho X + \beta}, \quad (18)$$

where β is an integration constant yet to be determined. We see from (15), (18), and the range of X and Y that

$$\sqrt{\beta - 2\rho A} \leq r \leq \sqrt{\beta + 2\rho A} \quad (19)$$

and

$$-B/\rho \leq \theta \leq B/\rho. \quad (20)$$

We denote the inner and outer radii of the deformed sector by

$$r_1 \equiv \sqrt{\beta - 2\rho A}, \quad r_2 \equiv \sqrt{\beta + 2\rho A}, \quad (21)$$

respectively. To ensure that r_1 is real, the constant of integration β must satisfy

$$\beta > 2\rho A. \quad (22)$$

From (16) we find the left Cauchy-Green tensor to be

$$\mathbf{B} = \begin{pmatrix} \rho^2/r^2 & 0 & 0 \\ 0 & r^2/\rho^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (23)$$

with principal invariants

$$I_1 = I_2 = \rho^2/r^2 + r^2/\rho^2 + 1, \quad I_3 = 1. \quad (24)$$

The inverse of \mathbf{B} is

$$\mathbf{B}^{-1} = \begin{pmatrix} r^2/\rho^2 & 0 & 0 \\ 0 & \rho^2/r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (25)$$

On using the constitutive law (2) for the Cauchy stresses in an isotropic, incompressible, hyperelastic material with strain energy density $W = W(I_1, I_2)$, we find that

$$T_{rr} = -p + 2(\rho^2/r^2)W_1 - 2(r^2/\rho^2)W_2, \quad (26)$$

$$T_{\theta\theta} = -p + 2(r^2/\rho^2)W_1 - 2(\rho^2/r^2)W_2, \quad (27)$$

$$T_{zz} = -p + 2W_1 - 2W_2, \quad (28)$$

$$T_{r\theta} = T_{rz} = T_{z\theta} = 0, \quad (29)$$

where

$$W_i \equiv \frac{\partial W}{\partial I_i} \Big|_{I_1=I_2=\rho^2/r^2+r^2/\rho^2+1} \quad (i=1,2), \quad (30)$$

and p is the, as yet undetermined, hydrostatic pressure.

The equilibrium equations in absence of body forces reduce to

$$\frac{\partial T_{rr}}{\partial r} + \frac{1}{r}(T_{rr} - T_{\theta\theta}) = 0, \quad \frac{\partial T_{\theta\theta}}{\partial \theta} = 0, \quad \frac{\partial T_{zz}}{\partial z} = 0. \quad (31)$$

Since W_i is a function of r only, we see from (26)-(28) and (31) that the hydrostatic pressure p must be a function of r only. Furthermore, we note that

$$\begin{aligned} \frac{dW}{dr} &= \frac{\partial W}{\partial I_1} \frac{dI_1}{dr} + \frac{\partial W}{\partial I_2} \frac{dI_2}{dr} \\ &= 2 \left(\frac{-\rho^2}{r^3} + \frac{r}{\rho^2} \right) \left(\frac{\partial W}{\partial I_1} + \frac{\partial W}{\partial I_2} \right) = -\frac{1}{r} (T_{rr} - T_{\theta\theta}), \end{aligned} \quad (32)$$

and so (31)₁ may be rewritten as

$$\frac{dT_{rr}}{dr} - \frac{dW}{dr} = 0. \quad (33)$$

Thus we find that

$$T_{rr}(r) = W(r) + K, \quad (34)$$

where K is a constant of integration.

We may now determine the integration constants K and β by applying the boundary conditions of traction-free lateral surfaces so that

$$T_{rr} = 0 \quad \text{at } r = r_1 \text{ and } r = r_2. \quad (35)$$

On using (34), the conditions (35) yield

$$W(r_1) + K = 0, \quad W(r_2) + K = 0, \quad (36)$$

so that

$$K = -W(r_1), \quad W(r_1) = W(r_2). \quad (37)$$

If we define

$$I(r) \equiv I_1(r) = I_2(r) = \rho^2/r^2 + r^2/\rho^2 + 1, \quad (38)$$

then we may write $W = W(I(r))$. Since one expects W to increase with increasing strain, we assume that W is a monotonically increasing function of I . Thus (37)₂ implies that

$$I(r_1) = I(r_2), \quad (39)$$

which may be rewritten as

$$\frac{\rho^2}{r_1^2} + \frac{r_1^2}{\rho^2} + 1 = \frac{\rho^2}{r_2^2} + \frac{r_2^2}{\rho^2} + 1. \quad (40)$$

This implies that

$$\rho^2 = r_1 r_2. \quad (41)$$

From (41) we see that ρ is the geometric mean of the inner and outer radii of the deformed beam and so $1/\rho$ may be thought of as the geometric mean curvature of the beam. Furthermore, $r = \rho$ is the location of the *neutral axis* of the deformed bar. This follows from that fact that vertical lines in the undeformed bar have length $L = 2B$, while from Fig. 1 it is seen that the image of these lines in the deformed configuration have length $l = 2Br/\rho$, and so the line at $r = \rho$ does not change length. The *bending angle* is given as

$$2\omega = l/r = 2B/\rho. \quad (42)$$

It can be easily shown that $I(r)$ has a *minimum* at $r = \rho$ where $I(\rho) = 3$ and so $W(r)$ is also a minimum there and $W(\rho) = 0$. Furthermore, it can be shown that I_{\max} , and therefore W_{\max} , occurs at $r = r_1$ and $r = r_2$. We can also see from (26)-(29) that at $r = \rho$ the stress field is a state of hydrostatic pressure with $T_{rr}(\rho) = T_{\theta\theta}(\rho) = T_{zz}(\rho) = -p(\rho) + 2(W_1(\rho) - W_2(\rho))$. (43) On using (21), equation (41) may be solved for β to yield

$$\beta = \sqrt{\rho^4 + 4\rho^2 A^2}, \quad (44)$$

where the positive square root is taken to ensure that (22) holds. On using (44) in (18) we find that $r = \rho$ corresponds to

$$X_n = \left(\rho - \sqrt{\rho^2 + 4A^2} \right) / 2. \quad (45)$$

Note that this result holds for *all* isotropic incompressible materials regardless of the strain-energy density employed and so is a *universal* (kinematic) relation. (See, e.g., Saccomandi, 2001, for a review of such results). As $\rho \rightarrow \infty$ in (45) (i.e. small bending), $X_n \rightarrow 0$ as one might anticipate since this corresponds to the geometric center-line of the undeformed beam. On the other hand, as $\rho \rightarrow 0$ (i.e. large bending) one finds from (45) that $X_n \rightarrow -A$ so that the neutral axis approaches the left hand side of the undeformed bar in this limit.

On substituting from (37) into (34) we find that

$$T_{rr}(r) = W(r) - W(r_1) \quad (46)$$

where r_1 is defined by (21) and (44). Since W is a maximum at r_1 , we see that T_{rr} is always *compressive*. Also, since $W = 0$ at $r = \rho$, we see that T_{rr} is a minimum on the neutral axis with value $T_{rr}(\rho) = -W(r_1) = -W(r_2)$. On returning to (31)₁ and making use of (46) we find that

$$T_{\theta\theta} = r \frac{dT_{rr}}{dr} + T_{rr} = \frac{d}{dr}(rT_{rr}) = \frac{d}{dr}[rW(r)] - W(r_1). \quad (47)$$

It may be shown on using (47) and the properties of W just discussed that $T_{\theta\theta}$ is *compressive* in a region near the inner surface and *tensile* near the outer surface, as one would expect physically. On using (26) and (46) we find that the hydrostatic pressure is given by

$$p = 2 \frac{\rho^2}{r^2} W_1 - 2 \frac{r^2}{\rho^2} W_2 - W(r) + W(r_1). \quad (48)$$

Thus, on the neutral axis, we find that (43) reduces to

$$T_{rr}(\rho) = T_{\theta\theta}(\rho) = T_{zz}(\rho) = -W(r_1), \quad (49)$$

and so all these stresses are *compressive* there. We find from (28) and (48) that the out of plane normal stress is

$$T_{zz} = 2 \left(1 - \frac{\rho^2}{r^2} \right) W_1 - 2 \left(1 - \frac{r^2}{\rho^2} \right) W_2 + W(r) - W(r_1). \quad (50)$$

Note that the resultant normal force on the end planes of the beam (i.e. at $\theta = \pm B/\rho$) is identically equal to zero since

$$\int_{r_1}^{r_2} T_{\theta\theta} dr = [rW(r) - rW(r_1)]_{r_1}^{r_2} = r_2 T_{rr}(r_2) - r_1 T_{rr}(r_1) = 0. \quad (51)$$

The resultant flexural couple on the end planes, per unit thickness of the beam, is given by

$$M = \int_{r_1}^{r_2} r T_{\theta\theta} dr = \int_{r_1}^{r_2} r \left\{ \frac{d}{dr}[rW(r)] - W(r_1) \right\} dr, \quad (52)$$

which upon integrating by parts and making use of (21) may be written as

$$M = 2\rho A W(r_1) - \int_{r_1}^{r_2} r W(r) dr. \quad (53)$$

Finally, we find the out-of-plane resultant force as

$$F_z = \int_{-B/\rho}^{B/\rho} \int_{r_1}^{r_2} T_{zz} r dr d\theta = \frac{2B}{\rho} \int_{r_1}^{r_2} T_{zz} r dr. \quad (54)$$

This force is necessary to maintain a state of plane strain.

We observe that T_{rr} depends only on $W(r)$ and that $T_{\theta\theta}$ and M depend only on $W(r)$ and dW/dr . However, T_{zz} and F_z involve derivatives of W with respect to the invariants.

4. Stress responses for classical constitutive models

In Rivlin (1949a,b), some results were given for the Mooney-Rivlin and neo-Hookean models. In the notation used in the present paper, for the Mooney-Rivlin material with strain-energy density given by (3), we find from (24) that

$$W^{MR}(r) = \frac{\mu}{2} \left(\frac{\rho^2}{r^2} + \frac{r^2}{\rho^2} - 2 \right). \quad (55)$$

From (46), (47), (50), (53), and (54) we find that

$$T_{rr}^{MR} = \frac{\mu}{2} \left(\frac{\rho^2}{r^2} + \frac{r^2}{\rho^2} - \frac{r_2}{r_1} - \frac{r_1}{r_2} \right), \quad (56)$$

$$T_{\theta\theta}^{MR} = \frac{\mu}{2} \left(-\frac{\rho^2}{r^2} + 3\frac{r^2}{\rho^2} - \frac{r_2}{r_1} - \frac{r_1}{r_2} \right), \quad (57)$$

$$T_{zz}^{MR} = \frac{\mu}{2} \left[(4\alpha - 2) + (1 - 2\alpha) \frac{\rho^2}{r^2} + (3 - 2\alpha) \frac{r^2}{\rho^2} - \frac{r_2}{r_1} - \frac{r_1}{r_2} \right], \quad (58)$$

$$M^{MR} = \frac{\mu}{2} \left(2A \sqrt{\rho^2 + (2A)^2} - \rho^2 \ln \frac{r_2}{r_1} \right), \quad (59)$$

and

$$F_z^{MR} = \mu AB(2\alpha - 1) \left[4 - 2\sqrt{1 + 4(A/\rho)^2} - \frac{\rho}{A} \ln \frac{r_2}{r_1} \right]. \quad (60)$$

We observe that the parameter α appearing in the definition (3) of the Mooney-Rivlin material appears only in (58) and (60) so that, except for the normal stress T_{zz} and its resultant force F_z , the stress distributions for the Mooney-Rivlin and neo-Hookean models are identical. This is a reflection of the comments made in the concluding paragraph of Section 3. On setting $\alpha = 1$ in (58) and (60), we thus find for a neo-Hookean material with strain-energy density given by (4) that

$$W^{nH}(r) = W^{MR}(r), \quad T_r^{nH} = T_r^{MR}, \quad T_{\theta\theta}^{nH} = T_{\theta\theta}^{MR}, \quad M^{nH} = M^{MR}, \quad (61)$$

$$T_{zz}^{nH} = \frac{\mu}{2} \left(2 - \frac{\rho^2}{r^2} + \frac{r^2}{\rho^2} - \frac{r_2}{r_1} - \frac{r_1}{r_2} \right), \quad (62)$$

and

$$F_z^{nH} = \mu AB \left[4 - 2\sqrt{1 + 4(A/\rho)^2} - \frac{\rho}{A} \ln \frac{r_2}{r_1} \right]. \quad (63)$$

On using (58) and (62), it can be shown that (for $\alpha \neq 1$) the out-of-plane stress is such that

$$T_{zz}^{MR} \geq T_{zz}^{nH}, \quad (64)$$

with equality only on the neutral axis where both stresses are *compressive* (see (49)). Both of these stresses are *compressive* in a region near the inner surface and *tensile* near the outer surface. Moreover, the resultant out-of-plane force F_z^{nH} given in (63) can be shown to be *compressive*. The situation is different for the Mooney-Rivlin model where F_z^{MR} is given by (60). For $1/2 < \alpha < 1$, this force is *compressive* while for $0 < \alpha < 1/2$, it is *tensile*. For the special case $\alpha = 1/2$, we see from (60) that

$$F_z^{MR} = 0. \quad (65)$$

Thus the character of the resultant out-of-plane force necessary to maintain plane strain for bending of a beam composed of a Mooney-Rivlin material changes from *compressive* to *tensile* as the material parameter α decreases and transitions through $\alpha = 1/2$.

5. Stress responses for strain-stiffening constitutive models

For the Gent model with strain-energy density given by (6), we find from (24) that

$$W^G(r) = -\frac{\mu}{2} J_m \ln \left\{ \left[J_m - \left(\frac{\rho^2}{r^2} + \frac{r^2}{\rho^2} - 2 \right) \right] / J_m \right\}. \quad (66)$$

In a quasi-static bending process, the constraint (6)₂ will first be reached on the inner and outer radii of the

deformed bar since we have $I_{\max} = I(r_1) = I(r_2)$.

Thus, to ensure that the pointwise constraint (6)₂ holds for *all* r in the range (20)₁, it is sufficient to assume the *global constraint*

$$r_1/r_2 + r_2/r_1 - 2 < J_m, \quad (67)$$

where we have made use of the relations (24) and (41). On using (21) and (44), the constraint (67) may be solved for A/ρ to yield

$$A/\rho < \sqrt{J_m^2 + 4J_m}/4. \quad (68)$$

For a given beam width (i.e. given A), the inequality (68) imposes an *upper limit on the amount of bending that a strain-stiffening beam composed of a Gent material can sustain*. In terms of the bending angle 2ω defined in (42), we write (68) as

$$\omega^G < \frac{B}{A} \frac{\sqrt{J_m^2 + 4J_m}}{4}. \quad (69)$$

Thus, for a given aspect ratio B/A of the beam and a given extensibility parameter J_m in the Gent model, the result (69) provides an *explicit expression for the maximum bending angle of the beam*.

From (46), (47), (50), and (53) we find that

$$T_{rr}^G = -\frac{\mu}{2} J_m \ln \left[\frac{J_m - \left(\frac{\rho^2}{r^2} + \frac{r^2}{\rho^2} - 2 \right)}{J_m - \left(\frac{r_2}{r_1} + \frac{r_1}{r_2} - 2 \right)} \right], \quad (70)$$

$$T_{\omega\omega}^G = \frac{\mu}{2} J_m \left\{ \ln \left[\frac{J_m - \left(\frac{\rho^2}{r^2} + \frac{r^2}{\rho^2} - 2 \right)}{J_m - \left(\frac{r_2}{r_1} + \frac{r_1}{r_2} - 2 \right)} \right] + 2 \frac{\frac{\rho^2}{r^2} - \frac{r^2}{\rho^2}}{J_m - \left(\frac{\rho^2}{r^2} + \frac{r^2}{\rho^2} - 2 \right)} \right\}, \quad (71)$$

$$T_{zz}^G = \frac{\mu}{2} J_m \left\{ \ln \left[\frac{J_m - \left(\frac{\rho^2}{r^2} + \frac{r^2}{\rho^2} - 2 \right)}{J_m - \left(\frac{r_2}{r_1} + \frac{r_1}{r_2} - 2 \right)} \right] + 2 \frac{\frac{\rho^2}{r^2} - 1}{J_m - \left(\frac{\rho^2}{r^2} + \frac{r^2}{\rho^2} - 2 \right)} \right\}, \quad (72)$$

$$M^G = \frac{\mu}{4} J_m \left[\rho^2 J_2 \ln \left(\frac{J_1 + J_2 - 2\frac{r_1}{r_2}}{J_1 + J_2 - 2\frac{r_2}{r_1}} \right) + \rho^2 (J_1 - J_2) \ln \left(\frac{r_1}{r_2} \right) - 4\rho A \right], \quad (73)$$

where

$$J_1 \equiv J_m + 2 \quad \text{and} \quad J_2 \equiv \sqrt{J_m(J_m + 4)}. \quad (74)$$

As $J_m \rightarrow \infty$ it can be verified on using l'Hopital's rule that these results reduce to (61)₂-(62) for the neo-Hookean model.

For the W^N model with strain-energy density given by (9), we find from (24) that

$$W^N(r) = -\frac{\mu(J-1)^2}{2J} \ln \left\{ \left[J^2 - \left(\frac{\rho^2}{r^2} + \frac{r^2}{\rho^2} \right) J + 1 \right] / (J-1)^2 \right\}. \quad (75)$$

For the bending deformation at hand (24)₁ shows that $I_1 = I_2$ and so by virtue of (12), the expression (75) should coincide with (66) if we formally replace J_m with \tilde{J} (defined in (13)). It may be verified directly that this is the case. The constraint (8)₂ for the W^N model can thus be deduced directly from (68) as

$$\frac{A}{\rho} < \frac{\sqrt{\tilde{J}^2 + 4\tilde{J}}}{4} = \frac{J^2 - 1}{4J}. \quad (76)$$

In terms of the bending angle ω , we get

$$\omega^N < \frac{B}{A} \left(\frac{J^2 - 1}{4J} \right). \quad (77)$$

From (46), (47), (50) and (53) we find that

$$T_{rr}^N = -\frac{\mu}{2} \tilde{J} \ln \left[\frac{\tilde{J} - \left(\frac{\rho^2}{r^2} + \frac{r^2}{\rho^2} - 2 \right)}{\tilde{J} - \left(\frac{r_2}{r_1} + \frac{r_1}{r_2} - 2 \right)} \right], \quad (78)$$

$$T_{\omega\omega}^N = -\frac{\mu}{2} \tilde{J} \left\{ \ln \left[\frac{\tilde{J} - \left(\frac{\rho^2}{r^2} + \frac{r^2}{\rho^2} - 2 \right)}{\tilde{J} - \left(\frac{r_2}{r_1} + \frac{r_1}{r_2} - 2 \right)} \right] + 2 \frac{\frac{\rho^2}{r^2} - \frac{r^2}{\rho^2}}{\tilde{J} - \left(\frac{\rho^2}{r^2} + \frac{r^2}{\rho^2} - 2 \right)} \right\}, \quad (79)$$

$$T_{zz}^N = \frac{\mu}{2} \tilde{J} \ln \left[\frac{\tilde{J} - \left(\frac{\rho^2}{r^2} + \frac{r^2}{\rho^2} - 2 \right)}{\tilde{J} - \left(\frac{r_2}{r_1} + \frac{r_1}{r_2} - 2 \right)} \right] - 2 \frac{(J-1) \left(J - \frac{r^2}{\rho^2} \right) \left(1 - \frac{\rho^2}{r^2} \right)}{J^2 - J \left(\frac{\rho^2}{r^2} + \frac{r^2}{\rho^2} \right) + 1}, \quad (80)$$

$$M^N = \frac{\mu(J-1)^2}{4J} \left[\rho^2 \frac{J^2 - 1}{J} \ln \left[\frac{J - \frac{r_1}{r_2}}{J - \frac{r_2}{r_1}} \right] + \rho^2 \frac{2}{J} \ln \left(\frac{r_1}{r_2} \right) - 4\rho A \right]. \quad (81)$$

In view of the remark made after (75) and the concluding paragraph of Section 3, we note that, except for the normal stress T_{zz} , the stress distributions and resultant moment for the Gent and W^N models are identical on formally replacing J_m with \tilde{J} (defined in (13)).

For a material with strain-energy density given by the Fung exponential model (14) we obtain from (24) that

$$W^F(r) = \frac{\mu}{2b} \left[e^{b \left(\frac{\rho^2}{r^2} + \frac{r^2}{\rho^2} - 2 \right)} - 1 \right], \quad (82)$$

and we find the Cauchy stresses from (46), (47), and (50) as

$$T_{rr}^F = \frac{\mu}{2b} \left[e^{b \left(\frac{\rho^2}{r^2} + \frac{r^2}{\rho^2} - 2 \right)} - e^{b \left(\frac{r_2}{r_1} + \frac{r_1}{r_2} - 2 \right)} \right], \quad (83)$$

$$T_{\omega\omega}^F = \frac{\mu}{2b} \left\{ \left[2b \left(-\frac{\rho^2}{r^2} + \frac{r^2}{\rho^2} \right) + 1 \right] e^{b \left(\frac{\rho^2}{r^2} + \frac{r^2}{\rho^2} - 2 \right)} - e^{b \left(\frac{r_2}{r_1} + \frac{r_1}{r_2} - 2 \right)} \right\}, \quad (84)$$

$$T_{zz}^F = \frac{\mu}{2b} \left\{ \left[2b \left(1 - \frac{\rho^2}{r^2} \right) + 1 \right] e^{b \left(\frac{\rho^2}{r^2} + \frac{r^2}{\rho^2} - 2 \right)} - e^{b \left(\frac{r_2}{r_1} + \frac{r_1}{r_2} - 2 \right)} \right\}. \quad (85)$$

6. Discussion

In Fig. 2 we plot the non-dimensionalized applied bending moment $M/\mu A^2$ versus the non-dimensionalized geometric mean curvature of the beam A/ρ ($=\omega(A/B)$) for the Gent model for the three representative values $J_m = 2.289, 30, 97.2$ and for $J_m = \infty$ which corresponds to both the Mooney-Rivlin and neo-Hookean models. The value $J_m = 97.2$ was suggested by Gent (1996) on the basis of experiments on rubber, while the value $J_m = 2.289$ was proposed by Horgan and Saccomandi (2003) as appropriate for human arterial wall tissue. The value $J_m = 30$ was chosen as an intermediate value. These four values correspond, by virtue of (13), to the W^N model with $J = 4.402, 32, 99.2$ and ∞ , respectively. The constraint (68) or (76) shows that these curves have vertical asymptotes (not shown in the figure) at $A/\rho = 0.949, 7.98,$ and 24.8 for the finite values of J_m (or J) respectively. The corresponding maximum bending angles are $\omega = 0.949(B/A), 7.98(B/A),$ and $24.8(B/A)$. For the Mooney-Rivlin and neo-Hookean models, the non-dimensionalized applied bending moment approaches a *horizontal asymptote* since it is easily shown from (59) or that

$$\lim_{A/\rho \rightarrow \infty} \frac{M^{MR/nH}}{\mu A^2} = 2. \quad (86)$$

This result does not appear to have been previously noticed in the literature for the classical models. The dotted lines in Fig. 2 are numerical results for the Fung exponential model with $b = .55, .035,$ and $.01$. These values of b were chosen to give results as similar as possible to the Gent and W^N models. We see from Fig. 2 that the Fung model gives results nearly identical to the Gent and W^N models up to relatively large bending, but the curves ultimately diverge, as they must, since the Fung model does *not* lead to a vertical asymptote.

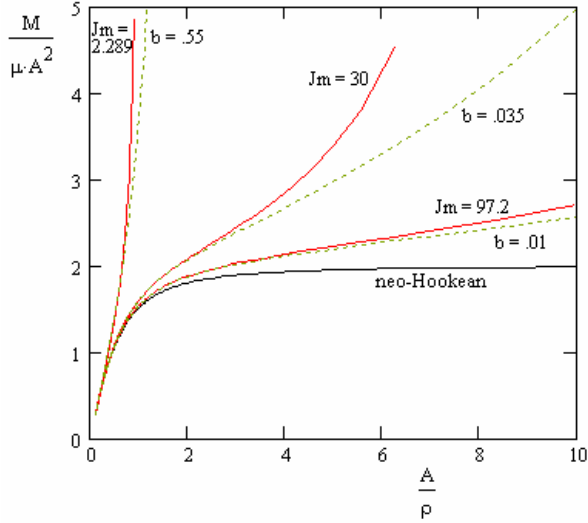


Fig. 2. Bending moment versus non-dimensionalized geometric mean curvature of the beam for the Gent model with $J_m = 2.289, 30, 97.2$ and ∞ , which is also valid for the W^N model with $J = 4.042, 32, 99.2$, and ∞ . The value $J_m = \infty$ corresponds to both the Mooney-Rivlin and neo-Hookean models. The dotted curves show numerical results for the Fung exponential model with $b = .55, .035$, and $.01$.

In Figs. 3-6 we plot the nondimensionalized stresses at a representative amount of curvature $A/\rho = \pi/4 \approx .79$ (i.e. $\rho/A \approx 1.3$). The value $A/\rho = \pi/4$ was chosen so that a bar with an aspect ratio of $B/A = 2$ would be deformed into a semi-circle (i.e. $2\omega = 2(B/A)(A/\rho) = \pi$). In Fig. 3 we plot T_{rr}/μ versus X/A , while in Fig. 4 we plot the same stress versus r/A , for the Gent and W^N models for the two smaller values of J_m and J and for the neo-Hookean model ($J_m = J = \infty$). The range for r/A in Fig. 4 is $r_1/A = 0.687 \leq r/A \leq 2.359 = r_2/A$. Results for the exponential model with $b = .035$ and $.55$ (dotted lines) are also shown for comparison purposes. For the smaller value of b , the curves are virtually coincident with those for the Gent model with $J_m = 30$. In Fig. 5 we plot the bending stress $T_{\theta\theta}/\mu$ versus X/A for the smallest value of J_m or J and for the neo-Hookean model ($J_m = J = \infty$) together with the corresponding result for the exponential model. The results for the latter model virtually coincide with those for the Gent and W^N models except near the right hand side of the undeformed beam. In Fig. 6, the out-of-plane stress T_{zz}/μ is plotted versus X/A for the Mooney-Rivlin model with $\alpha = 0.2, 0.4, 0.6, 0.8$, and 1 , where the last value corresponds to the neo-Hookean case. This plot

illustrates the analytic result (64), i.e. $T_{zz}^{MR} \geq T_{zz}^{nH}$ and shows that these stresses are compressive near the inner surface and tensile near the outer. In Fig. 7 the corresponding plot is given for the Gent and W^N models with $J_m = 2.289$ and $J = 4.042$, respectively, for the Fung model with $b = .55$, and for the neo-Hookean model.

Fig. 4 illustrates features of the compressive stress T_{rr} discussed in Section 3 for general strain-energies. All of the curves in Fig. 4 are seen to reach their minima on the neutral axis where $r/A = \rho/A = 1.3$. Equivalently, on using (45), the neutral axis is located at $X/A = -.55$ in the undeformed bar (see Fig. 3). It can also be seen that the stress values at these minima decrease as the stiffening parameters decrease. In fact, at any given value of r/A , the stresses T_{rr} decrease monotonically with J_m or J .

The bending stress $T_{\theta\theta}$ plotted in Fig. 5 is compressive near the inner surface of the bent beam and tensile near the outer surface. We observe that the magnitude of the bending stress for the strain-stiffening models is much larger than that for the neo-Hookean model, especially near the inner and outer surfaces.

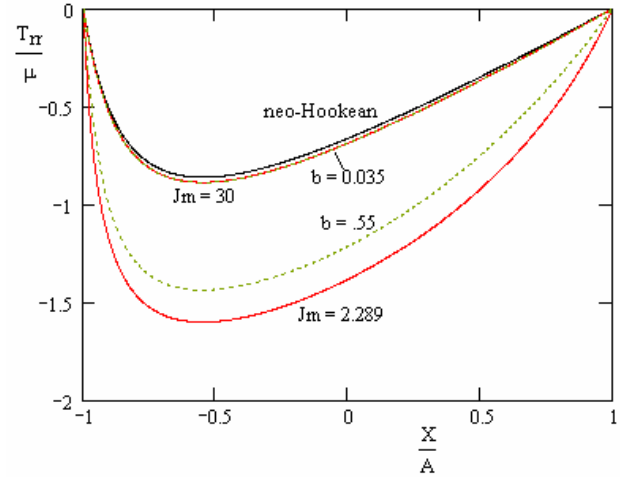


Fig. 3. T_{rr}/μ versus X/A with $A/\rho = \pi/4 \approx .79$ for the Gent model with $J_m = 2.289, 30$, and ∞ , which is also valid for the W^N model with $J = 4.042, 32$, and ∞ . The value $J_m = \infty$ corresponds to both the Mooney-Rivlin and neo-Hookean models. The dotted curves show results for the Fung exponential model with $b = .035$ and $.55$. For the smaller value of b , this curve is virtually coincident with that for the Gent model with $J_m = 30$.

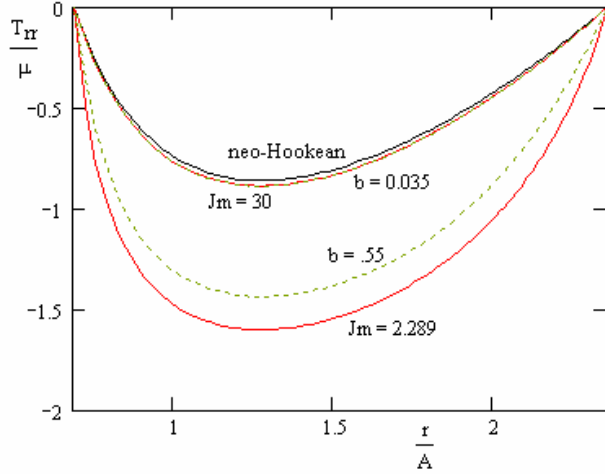


Fig. 4. T_{rr}/μ versus r/A with $A/\rho = \pi/4 \approx .79$ for the Gent model with $J_m = 2.289, 30$, and ∞ , which is also valid for the W^N model with $J = 4.402, 32$, and ∞ . The value $J_m = \infty$ corresponds to both the Mooney-Rivlin and neo-Hookean models. The dotted curves show results for the Fung exponential model with $b = .035$ and $.55$. For the smaller value of b , this curve is virtually coincident with that for the Gent model with $J_m = 30$.

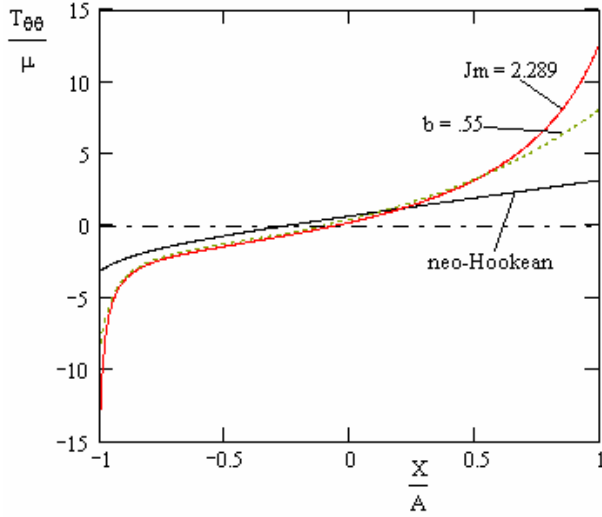


Fig. 5. $T_{\theta\theta}/\mu$ versus X/A with $A/\rho = \pi/4 \approx .79$ for the Gent model with $J_m = 2.289$ and ∞ , which is also valid for the W^N model with $J = 4.402$ and ∞ . The value $J_m = \infty$ corresponds to both the Mooney-Rivlin and neo-Hookean models. The dotted curve shows results for the Fung exponential model with $b = .55$, which virtually coincides with that for the Gent and W^N models except near the right side of the undeformed beam.

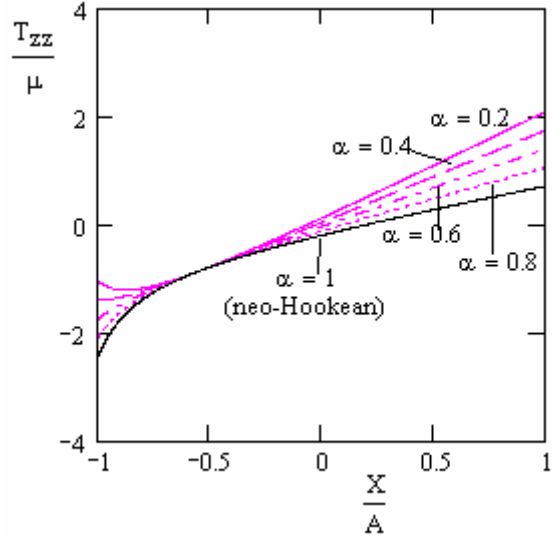


Fig. 6. T_{zz}/μ versus X/A with $A/\rho = \pi/4 \approx .79$ for the Mooney-Rivlin model with $\alpha = 0.2, 0.4, 0.6, 0.8$, and 1.0 .

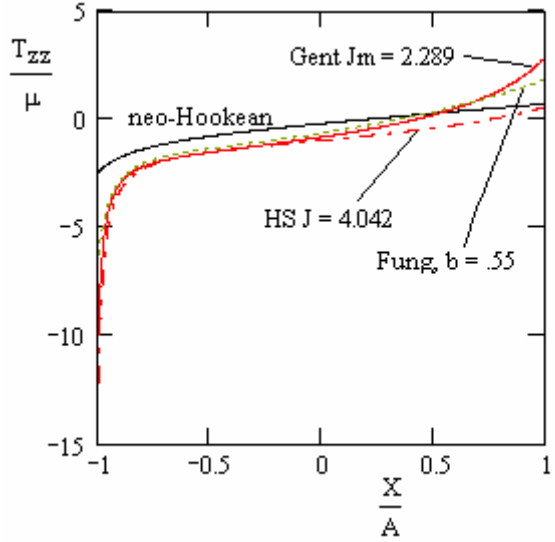


Fig. 7. T_{zz}/μ versus X/A with $A/\rho = \pi/4 \approx .79$ for the Gent model (solid lines) with $J_m = 2.289$ and ∞ , for the W^N model (dashed line) with $J = 4.402$ and ∞ , and for the Fung exponential model with $b = .55$. The values $J_m = \infty$ and $J = \infty$ correspond to the neo-Hookean model.

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