ELECTROMAGNETIC SCATTERING FROM PERTURBED SURFACES

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Abstract

This paper is concerned with the study of scattering of electromagnetic waves from a (local) perturbation of a fixed surface, the boundary of a given obstacle in $\mathbb{R}^3$. The goal is to produce an algorithm for solving boundary value problems in the exterior of the perturbed domain solely based on the knowledge of the Green function for the original surface. This is done by solving a boundary integral equation which only involves the perturbed portion of the boundary.

1. Introduction

This paper deals with the case of electromagnetic scattering from a perturbation of a fixed surface that is the boundary of a given, solid obstacle in three dimensions, such as the fuselage of an airplane. The perturbation under consideration consists of finitely many bumps. We reduce an exterior boundary value problem involving Maxwells system of equations to solving a boundary integral equation where the integration is carried out only over the bumps (and not over the entire surface). Maxwells equations model the interaction between electromagnetic fields and matter.

Consider an electromagnetic wave in $\mathbb{R}^3$ composed of two parts, an electric wave and a magnetic wave, which are perpendicular to one another. Call the electric portion of the wave $\vec{E} = (E_1, E_2, E_3)$ and the magnetic portion $\vec{H} = (H_1, H_2, H_3)$. Let $\Omega \subseteq \mathbb{R}^3$, with boundary surface $\partial\Omega$, be an object which the electromagnetic wave is propagating toward. Writing Maxwell's equations, we have

$$\begin{cases} \operatorname{curl}\vec{E} - i\,k\,\vec{H} = \vec{0} \quad \text{in } \mathbb{R}^3 \setminus \overline{\Omega}, \\ \operatorname{curl}\vec{H} + i\,k\,\vec{E} = \vec{0} \quad \text{in } \mathbb{R}^3 \setminus \overline{\Omega}, \\ \vec{n} \times \vec{E}\Big|_{\partial\Omega} = \vec{g} \in L^2_{\tan}(\partial\Omega), \\ \vec{E}, \vec{H} \;\; \text{satisfy the Silver-Muller} \\ \quad \text{radiation condition at infinity,} \end{cases} \tag{1.1}$$

where we refer the reader to e.g., p. 153 in [1], for details on the radiation condition. Above $\vec{n} = (n_1, n_2, n_3)$ is the outward unit normal vector to $\partial\Omega$, $\vec{g}$ is a given tangential field defined on $\partial\Omega$ with $L^2(\partial\Omega)$ components, and $k \neq 0$ is the wave number associated with the electromagnetic wave. In general, curl is the standard curl operator, i.e., for a smooth vector $\vec{F}$ in Ω we have

$$\operatorname{curl}\vec{F} := \nabla \times \vec{F}, \tag{1.2}$$

where $\times$ stands for the cross product of vectors and $\nabla = (\partial_1, \partial_2, \partial_3)$ is the usual gradient.

A simple examination of (1.1) together with the fact that

$$\operatorname{curl}(\operatorname{curl}\vec{E}) = -\Delta\vec{E} + \nabla\operatorname{div}\vec{E} \tag{1.3}$$

gives

$$\begin{aligned} &\operatorname{curl}\left(\tfrac{1}{ik}\operatorname{curl}\vec{E}\right) + ik\vec{E} = \\ &\tfrac{1}{ik}\left(\operatorname{curl}(\operatorname{curl}\vec{E}) - k^2\vec{E}\right) = \\ &-\tfrac{1}{ik}(\Delta + k^2)\vec{E} + \tfrac{1}{ik}\nabla\operatorname{div}\vec{E} = \vec{0}, \end{aligned} \tag{1.4}$$

where Δ is the Laplacian (acting on vectors in a component-wise fashion), and div is the standard divergence operator. Also, we notice that, since $\operatorname{div}(\operatorname{curl}\vec{F}) = 0$ for any smooth vector $\vec{F}$ in Ω, we have that

$$\operatorname{div}(\operatorname{curl}\vec{H}) + ik\,\operatorname{div}\vec{E} = 0 \;\Rightarrow\; \operatorname{div}\vec{E} = 0. \tag{1.5}$$

Thus, (1.1) together with (1.4) and (1.5), lead us to considering the following boundary value problem

$$\begin{cases} (\Delta + k^2)\vec{E} \;=\; \vec{0} \quad \text{in } \mathbb{R}^3 \setminus \overline{\Omega}, \\ \operatorname{div}\vec{E} = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{\Omega}, \\ \vec{n} \times \vec{E}\Big|_{\partial\Omega} = \vec{g} \in L^2_{\tan}(\partial\Omega), \\ \vec{E} \;\text{ satisfies the radiation condition} \\ \qquad \text{at infinity.} \end{cases} \tag{1.6}$$

This paper is concerned with the study of scattering of electromagnetic waves from a perturbation $\partial\Omega$ of a fixed surface $S = \partial\Omega^o$ (the boundary of a given, solid obstacle Ω^o in $\mathbb{R}^3$). The perturbation is assumed to be local in nature, in that it is confined to a finite portion – in fact, consisting of finitely many "bumps." Our main goal is to be able to solve (1.6) knowing only the Green's function for the original, unperturbed surface and the boundary values on the surface with the bumps. This type of result is highly desirable in the context of solving a large number of scattering problems in which the underlying surfaces are slight perturbations of a fixed shape. The reason is that, in general, the task of computing the Green function for a given domain is a challenging, delicate one. Our key technical achievement in this regard is the reduction of (1.6) to solving a boundary integral equation where the integration is carried out only over the bumps (and not over the entire surface). More specifically our main result reads as follows.

Theorem. *Consider Ω^o a bounded Lipschitz domain in $\mathbb{R}^3$ such that $\partial\Omega^o$ is the original surface partitioned as*

$$\partial\Omega^o = A \cup B^o, \tag{1.7}$$

where B^o is the portion of $\partial\Omega^o$ above which the bumps are added. Denoting by B the bumps which are assumed to lie above B^o, we set $\partial\Omega$ for the perturbed surface given by

$$\partial\Omega := A \cup B, \quad \textit{where} \quad \Omega^o \subseteq \Omega. \tag{1.8}$$

Consider the boundary value problem

$$\begin{cases} (\Delta + k^2)\vec{E} \;=\; \vec{0} \quad in\ \mathbb{R}^3 \setminus \overline{\Omega}, \\ \mathrm{div}\vec{E} = 0 \quad in\ \mathbb{R}^3 \setminus \overline{\Omega}, \\ \vec{n} \times \vec{E}\Big|_{\partial\Omega} = \vec{g} \in L^2_{\tan}(\partial\Omega), \\ \vec{E}\ \ satisfies\ the\ radiation\ condition \\ \qquad at\ infinity. \end{cases} \tag{1.9}$$

Then, in order to solve (1.9) knowing the Maxwell Green function Γ^o for the unperturbed surface Ω^o, one should take the following steps:

- *solve the boundary integral equation in the unknown $\vec{h} \in L^2_{\tan}(B)$ given by*

$$\begin{aligned} &-\tfrac{1}{2}\vec{h}(Z) + \vec{n}(Z)\times \\ &\qquad \int_B \big[\mathbf{\nabla}_Y \Gamma^o(Z,Y)\big]^t \vec{h}(Y)\, d\sigma(Y) \\ &= \vec{\mathcal{G}}(Z), \quad for\ a.\ e.\ Z \in B, \end{aligned} \tag{1.10}$$

where

$$\begin{aligned} &\vec{\mathcal{G}}(Z) := \vec{g}(Z) - \vec{n}(Z)\times \\ &\qquad \textstyle\int_A \big[\mathbf{\nabla}_Y \Gamma^o(Z,Y)\big]^t \vec{g}(Y)\, d\sigma(Y), \end{aligned} \tag{1.11}$$

and $\mathbf{\nabla}_Y$ is the 3×3 matrix of first order partial derivatives from (2.9) acting in the variable Y.

- *set* $\vec{f}(X) := \begin{cases} \vec{h}(X) & for \quad X \in B, \\ \vec{g}(X) & for \quad X \in A. \end{cases}$
- *let $u = \mathcal{Q}(\vec{f})$ where*

$$\mathcal{Q}(\vec{f})(X) := \int_{\partial\Omega} \big[\mathbf{\nabla}_Y \Gamma^o(X,Y)\big]^t \vec{f}(Y)\ d\sigma(Y), \tag{1.12}$$

where

$$X \in \mathbb{R}^3 \setminus \overline{\Omega}.$$

Then u is a solution of (1.9).

This type of result is significant for the numerical treatment of (1.6) via boundary element methods. In a subsequent publication, we plan to consider this numerical issue in greater detail.

The present work builds on the paper [3], where a similar algorithm for the reduced wave (Helmholtz) equation was devised. Here we take the natural next step of considering *systems* of PDE's. Among other things, working with vector-valued functions introduces new analytic and algebraic difficulties that have to be dealt with.

The layout of the paper is as follows. In §2 we introduce the relevant notation and review a number of preliminary results. The Green function associated with the Maxwell system is introduced in §3, where we also state its main properties. In §4 we define a solution operator associated to (1.6) and provide its boundary behavior. In particular we show how solving (1.6) is reduced to knowing the Green function of the original domain Ω^o and the solution of a 'local' integral equation over the added bumps. Proofs of results contained in §3 and §4 are to appear to in [5].

2. Preliminaries

In this section we introduce some basic notation and recall some known results used in the rest of the paper.

Definition 2.1. *A bounded domain $\Omega \subset \mathbb{R}^3$ is called Lipschitz if for any $X_0 \in \partial\Omega$ there exist r, $h > 0$ and a coordinate system*

$\{x_1, x_2, x_3\}$ *in* $\mathbb{R}^3$ *(isometric to the canonical one) with origin at* X_0 *along with a function* $\varphi : \mathbb{R}^2 \to \mathbb{R}$ *which is Lipschitz and so that the following holds. If* $C(r, h)$ *denotes the rectangle* $(-r, r) \times (-h, h)$ *in the new system of coordinates then*

$$\begin{aligned} \Omega \cap C(r,h) &= \{(x_1,x_2,x_3);\ |(x_1,x_2)| < r \\ &\qquad \textit{and } \varphi(x_1,x_2) < x_3 < h\}, \\ \partial\Omega \cap C(r,h) &= \{(x_1,x_2,x_3);\ |(x_1,x_2)| < r \\ &\qquad \textit{and } x_3 = \varphi(x_1,x_2)\}. \end{aligned} \tag{2.1}$$

Also, for a fixed, sufficiently large constant $\kappa > 1$, the non-tangential approach region corresponding to $X \in \partial\Omega$ is defined by

$$\Upsilon^-(X) := \{Y \in \mathbb{R}^3 \setminus \overline{\Omega};\ |X - Y| < \kappa \,\mathrm{dist}\,(Y, \partial\Omega)\}. \tag{2.2}$$

Throughout the paper all restrictions to the boundary are taken in the non-tangential sense, i.e.,

$$u|_{\partial\Omega}(X) := \lim_{\substack{X \in \Upsilon^-(P) \\ X \to P}} u(X), \text{for a.e. } P \in \partial\Omega, \tag{2.3}$$

whenever the limit exists. Going further, we let $d\sigma$ stand for the surface measure on $\partial\Omega$. Then the unit normal $\vec{n} = (n_1, n_2, n_3)$ to $\partial\Omega$ is well defined at almost every (with respect to $d\sigma$) point on $\partial\Omega$. In the sequel we also let $\langle\cdot,\cdot\rangle$ denote the canonical scalar product in $\mathbb{R}^3$.

Next, let $\vec{\mathcal{F}}$, $\vec{\mathcal{G}}$ be smooth vectors in $\Omega \subset \mathbb{R}^3$, where Ω is a bounded Lipschitz domain. Straightforward manipulations give that

$$\mathrm{curl}(\mathrm{curl}\vec{\mathcal{F}}) = -\Delta\vec{\mathcal{F}} + \nabla\mathrm{div}\vec{\mathcal{F}} \quad \text{in} \quad \Omega. \tag{2.4}$$

Also, via integration by parts, we can establish the following Green-type formulas:

$$\int_\Omega \langle \vec{\mathcal{F}}, \mathrm{curl}\ \vec{\mathcal{G}}\rangle\, dX = \int_\Omega \langle \mathrm{curl}\ \vec{\mathcal{F}}, \vec{\mathcal{G}}\rangle\, dX + \int_{\partial\Omega} \langle \vec{\mathcal{F}}, \vec{n} \times \vec{\mathcal{G}}\rangle\, d\sigma, \tag{2.5}$$

and

$$\begin{aligned} \int_\Omega \langle (\Delta + k^2)\vec{\mathcal{F}}, \vec{\mathcal{G}}\rangle - \langle \vec{\mathcal{F}}, (\Delta + k^2)\vec{\mathcal{G}}\rangle\, dX = \\ \int_{\partial\Omega} \langle \mathrm{curl}\ \vec{\mathcal{F}}, \vec{n} \times \vec{\mathcal{G}}\rangle + \langle \vec{\mathcal{G}}\,\mathrm{div}\vec{\mathcal{F}}, \vec{n}\rangle\, d\sigma \\ - \int_{\partial\Omega} \langle \mathrm{curl}\ \vec{\mathcal{G}}, \vec{n} \times \vec{\mathcal{F}}\rangle + \langle \vec{\mathcal{F}}\,\mathrm{div}\vec{\mathcal{G}}, \vec{n}\rangle\, d\sigma. \end{aligned} \tag{2.6}$$

Next, for any vector $\vec{\mathcal{F}}$ in $\mathbb{R}^3$, we have

$$\vec{n}(Y) \times \vec{\mathcal{F}} = N_Y \vec{\mathcal{F}}, \tag{2.7}$$

where

$$N_Y := \begin{pmatrix} 0 & -n_3(Y) & n_2(Y) \\ n_3(Y) & 0 & -n_1(Y) \\ -n_2(Y) & n_1(Y) & 0 \end{pmatrix}, \tag{2.8}$$

for almost every $Y \in \partial\Omega$. Finally, for any 3×3 matrix $A(Y)$ with differentiable entries we have

$$\mathrm{curl}_Y[A(Y)\vec{c}] = [\blacktriangledown_Y A(Y)]\vec{c}, \tag{2.9}$$

where

$$\blacktriangledown = \begin{pmatrix} 0 & -\partial_3 & \partial_2 \\ \partial_3 & 0 & -\partial_1 \\ -\partial_2 & \partial_1 & 0 \end{pmatrix}, \tag{2.10}$$

and the subscript Y denotes the variable with respect to which partial derivatives are taken.

We end this section by recording a result useful in the sequel. To state it, let $\Phi(X,Y)$ be the fundamental solution for the Helmholtz operator in $\mathbb{R}^3$. Then

$$(\Delta_Y + k^2)\Phi(X,Y) = \delta_X(Y), \qquad (2.11)$$

where δ_X is the delta-Dirac distribution with mass at X and Φ is given by

$$\Phi(X,Y) := \frac{e^{-k|X-Y|}}{4\pi\,|X-Y|}, X, Y \in \mathbb{R}^3, X \neq Y. \qquad (2.12)$$

Also, for a bounded Lipschitz domain $\Omega \subset \mathbb{R}^3$ we set

$$L^2_{\tan}(\partial\Omega) := \{\vec{f} = (f_1, f_2, f_3) : f_j \in L^2(\partial\Omega), \quad j = 1,2,3, \text{ and } \langle \vec{n}, \vec{f}\rangle = 0 \text{ a.e. on } \partial\Omega\}. \qquad (2.13)$$

We have

Theorem 2.2. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^3$, $\vec{f} \in L^2_{\tan}(\partial\Omega)$ and $\Phi(X,Y)$ be as in* (2.12)*. Then the following holds*

$$\lim_{\substack{X\in\Upsilon^-(Z)\\ X\to Z}} \int_{\partial\Omega} \nabla_X \Phi(X,Y) \times \vec{f}(Y)\, d\sigma(Y) = \frac{1}{2}\vec{n}(Z) \times \vec{f}(Z) + \int_{\partial\Omega} \nabla_Z \Phi(Z,Y) \times \vec{f}(Y)\, d\sigma(Y), \qquad (2.14)$$

for almost every $Z \in \partial\Omega$.

3. The Maxwell Green function

In this section we discuss the construction of the Maxwell Green function for the exterior of a bounded Lipschitz domain $\Omega^o \subset \mathbb{R}^3$ and some useful properties. To this end, for any $\vec{c}$ vector in $\mathbb{R}^3$ we introduce

$$\vec{G^o}(X,Y;\vec{c}) := \Phi(X,Y)\vec{c} - \vec{U^o}(X,Y;\vec{c}), \qquad (3.1)$$

where $\Phi(X,Y)$ is as in (2.12) and $\vec{U^o}(X,Y;\vec{c})$ satisfies

$$\begin{cases} (\Delta_Y + k^2)\vec{U^o}(X,Y;\vec{c}) = \vec{0} \text{ in } \mathbb{R}^3 \setminus \overline{\Omega^o}, \\ \operatorname{div}_Y \vec{U^o}(X,Y;\vec{c}) = \nabla_Y \Phi(X,Y)\cdot\vec{c} \\ \qquad \text{in } \ \mathbb{R}^3 \setminus \overline{\Omega^o}, \\ \vec{n}(Y) \times \vec{U^o}(X,Y;\vec{c})|_{Y\in\partial\Omega^o} = \\ \qquad \Phi(X,Y)\left(\vec{n}(Y)\times\vec{c}\right), \\ \vec{U^o}(X,\cdot;\vec{c}) \text{ satisfies the radiation} \\ \qquad \text{condition at infinity.} \end{cases} \qquad (3.2)$$

Employing Theorem 6.2 from p. 163 in [4] the above boundary problem has a unique solution. Also an inspection shows that for a fixed $\vec{c}$ constant vector in $\mathbb{R}^3$ and a fixed $Y \in \mathbb{R}^3 \setminus \Omega^o$, we have

$$\mathbb{R}^3 \setminus \overline{\Omega^o} \ni X \to \nabla_Y U^o(X,Y;\vec{c}) \qquad (3.3)$$

is continuous.

Therefore, based on (3.2) we infer that

$$\begin{cases} (\Delta_Y + k^2)\vec{G^o}(X,Y;\vec{c}) = \delta_X(Y)\vec{c} \\ \qquad \text{in } \ \mathbb{R}^3 \setminus \overline{\Omega^o}, \\ \operatorname{div}_Y \vec{G^o}(X,Y;\vec{c}) = 0 \ \text{ in } \ \mathbb{R}^3 \setminus \overline{\Omega^o}, \\ \vec{n}(Y) \times \vec{G^o}(X,Y;\vec{c})|_{Y\in\partial\Omega^o} = \vec{0}, \\ \vec{G^o}(X,\cdot;\vec{c}) \text{ satisfies the radiation} \\ \qquad \text{condition at infinity.} \end{cases} \qquad (3.4)$$

Next we claim that $\vec{G}^o(X,Y;\vec{c})$ depends linearly on $\vec{c}$. Hence for any fixed $X, Y \in \mathbb{R}^3 \setminus \Omega^o$, $X \neq Y$, there exists a 3×3 matrix $\Gamma^o(X,Y)$, called in the sequel the Green matrix-valued function for the Maxwell equations, such that

$$\vec{G}^o(X,Y;\vec{c}) = \Gamma^o(X,Y)\,\vec{c}, \ \forall\ \vec{c}\in\mathbb{R}^3\,. \quad (3.5)$$

Indeed, since the solution of the problem (3.2) depends linearly on the right hand sides which in turn are linear in $\vec{c}$ we conclude that $\vec{U}^o(X,Y;\vec{c})$ is linear in $\vec{c}$ and so is $\vec{G}^o(X,Y;\vec{c})$.

The following result is a collection of some useful properties of the Green matrix-valued function $\Gamma^o(X,Y)$ for the Maxwell equations. We have

Proposition 3.3. *Let $\Omega^o \subset \mathbb{R}^3$ be a bounded Lipschitz domain and $\Gamma^o(X,Y)$ be the Green matrix-valued function introduced in* (3.5). *Then, for each $X \in \mathbb{R}^3 \setminus \overline{\Omega^o}$, the following hold*

$$(\Delta_Y + k^2)\Gamma^o(X,Y) = \delta_X(Y) I_3 \quad in \quad \mathbb{R}^3\setminus\overline{\Omega^o}\,, \quad (3.6)$$

where I_3 denotes the 3×3 identity matrix.

Let $\Gamma^{o,j}(X,Y)$, $j = 1,2,3$, denote the j-th column of $\Gamma^o(X,Y)$. Then, for each $X \in \mathbb{R}^3\setminus\overline{\Omega^o}$ we have

$$\mathrm{div}_Y \Gamma^{o,j}(X,Y) = 0 \quad in \quad \mathbb{R}^3\setminus\overline{\Omega^o}, \quad (3.7)$$

and

$$N_Y \Gamma^o(X,Y)\Big|_{Y\in\partial\Omega^o} = \vec{0}, \quad (3.8)$$

where N_Y is the 3×3 matrix from (2.7).

The main result of this section is the following.

Theorem 3.4. *Let $\Omega^o \subset \mathbb{R}^3$ be a bounded Lipschitz domain and let $\Gamma^o(X,Y)$ be the Maxwell Green matrix-valued function introduced in* (3.5). *Then, for any $X_0 \neq X_1$ in $\mathbb{R}^3\setminus\overline{\Omega^o}$, we have*

$$\Gamma^o(X_1,X_0) = \left[\Gamma^o(X_0,X_1)\right]^t. \quad (3.9)$$

Also, if $\vec{E}$ solves the problem

$$\begin{cases} (\Delta + k^2)\vec{E} = \vec{0} \quad in \quad \mathbb{R}^3\setminus\overline{\Omega^o}, \\ \mathrm{div}\,\vec{E} = 0 \quad in \quad \mathbb{R}^3\setminus\overline{\Omega^o}, \\ \vec{n}\times\vec{E} = \vec{f} \in L^2_{\tan}(\partial\Omega^o), \\ \vec{E} \ \text{satisfies the radiation condition} \\ \qquad \text{at infinity,} \end{cases} \quad (3.10)$$

then

$$\vec{E}(X) = \int_{\partial\Omega^o} \left[\blacktriangledown_Y \Gamma^o(X,Y)\right]^t \vec{f}(Y)\, d\sigma(Y), \quad (3.11)$$

for

$$X \in \mathbb{R}^3\setminus\overline{\Omega^o}.$$

4. Scattering from perturbed surfaces

In this section we study the electromagnetic scattering from a perturbed surface. The perturbation is assumed to be local in nature, in that it is confined to a finite portion consisting of finitely many "bumps." Consider Ω^o a bounded Lipschitz domain in $\mathbb{R}^3$ such that $\partial\Omega^o$ is the original surface partitioned as

$$\partial\Omega^o = A \cup B^o, \quad (4.1)$$

where B^o is the portion of $\partial\Omega^o$ above which the bumps are added. Denoting by B the bumps which are assumed to lie above B^o, the perturbed surface of interest is

$$\partial\Omega := A\cup B, \quad \text{where} \quad \Omega^o \subseteq \Omega. \quad (4.2)$$

Our main result of this section states that one can solve the boundary value problem (1.6) knowing only the Green matrix-valued function for the Maxwell equations in the original domain Ω^o and the solution of a boundary integral equation where the integration is carried only over the bumps.

For $X, Y \in \mathbb{R}^3 \setminus \overline{\Omega^o}$, $X \neq Y$ and $\vec{c}$ a constant vector in $\mathbb{R}^3$, we let $G^o(X, Y; \vec{c}) = \Gamma^o(X, Y)\vec{c}$, where $\Gamma^o(X, Y)$ is the Green matrix-valued function for the Maxwell equations introduced in (3.5). According to Proposition 3.3 we have

$$\begin{cases} (\Delta_Y + k^2)\Gamma^o(X, Y) = \delta_X(Y) I_3 \\ \qquad \text{in } \mathbb{R}^3 \setminus \overline{\Omega^o}, \\ \operatorname{div}_Y \Gamma^{o,j}(X, Y) = 0 \text{ in } \mathbb{R}^3 \setminus \overline{\Omega^o}, \\ N_Y \Gamma^o(X, Y)\Big|_{Y \in \partial\Omega^o} = \vec{0}. \end{cases} \tag{4.3}$$

Above $\Gamma^{o,j}(X, Y)$ denotes the j-th column of $\Gamma^o(X, Y)$, $j = 1, 2, 3$, and N_Y is the matrix introduced in (2.7).

Next, we introduce the solution operator for the boundary value problem (1.6) in the perturbed domain Ω (defined in terms of the Green function of the original surface $\partial\Omega^o$) and state some if its trademark properties. Specifically, for each tangential vector field $\vec{f} \in L^2_{\tan}(\partial\Omega)$ we set

$$\mathcal{Q}(\vec{f})(X) := \int_{\partial\Omega} \big[\blacktriangledown_Y \Gamma^o(X, Y)\big]^t \vec{f}(Y)\, d\sigma(Y), \tag{4.4}$$

for

$$X \in \mathbb{R}^3 \setminus \overline{\Omega}.$$

then the following holds.

Theorem 4.5. *Let $\Omega \subset \Omega^o \subset \mathbb{R}^3$ be bounded Lipschitz domains such that* (4.1)-(4.2) *are satisfied. For each tangential vector field $\vec{f} \in L^2_{\tan}(\partial\Omega)$ we have*

$$(\Delta + k^2)\mathcal{Q}(\vec{f}) \equiv 0 \quad \text{in} \quad \mathbb{R}^3 \setminus \overline{\Omega}, \tag{4.5}$$

and

$$\operatorname{div}\mathcal{Q}(\vec{f}) \equiv 0 \quad \text{in} \quad \mathbb{R}^3 \setminus \overline{\Omega}. \tag{4.6}$$

Also, one can easily check that $\mathcal{Q}(\vec{f})$ satisfies the radiation condition at infinity.

Our next results concern the boundary behavior of the solution operator. We start with the following useful technical result.

Proposition 4.6. *Recall the bumps B and the B^o portion of the original surface over which these lay as in* (4.1)-(4.2) *and denote by $\mathcal{O}$ the sub-domain of Ω with the property $\partial\mathcal{O} = B \cup B^o$. Consider $\vec{V}$ a vector field in $\mathcal{O}$ such that*

$$\begin{cases} (\Delta + k^2)\vec{V} = \vec{0} \text{ in } \mathcal{O}, \\ \operatorname{div} \vec{V} = 0 \text{ in } \mathcal{O}. \end{cases} \tag{4.7}$$

Then, for each $X \in \mathbb{R}^3 \setminus \overline{\Omega}$ the following holds

$$\int_{\partial\mathcal{O}} \big[\blacktriangledown_Y \Gamma^o(X, Y)\big]^t (\vec{n}(Y) \times \vec{V}(Y))\, d\sigma(Y) = - \int_B \Gamma^o(Y, X)(\vec{n}(Y) \times \operatorname{curl}_Y \vec{V}(Y))\, d\sigma(Y). \tag{4.8}$$

Theorem 4.7. *Let $\Omega^o \subseteq \Omega$ be Lipschitz domains as in* (4.1)-(4.2) *and recall the boundary decomposition $\partial\Omega = A \cup B$. Then for each tangential field $\vec{f} \in L^2_{\tan}(\partial\Omega)$ we have*

$$\lim_{\substack{X \in \Upsilon^-(Z) \\ X \to Z}} \vec{n}(Z) \times \mathcal{Q}(\vec{f})(X) = \vec{f}(Z), \tag{4.9}$$

for

$$a.e. \quad Z \in A,$$

and

$$\lim_{\substack{X\in\Upsilon^-(Z)\\ X\to Z}} \vec{n}(Z)\times\mathcal{Q}(\vec{f})(X) = -\tfrac{1}{2}\vec{f}(Z)+\vec{n}(Z)\times \int_{\partial\Omega}\big[\blacktriangledown_Y\Gamma^o(Z,Y)\big]^t\vec{f}(Y)\,d\sigma(Y), \tag{4.10}$$

for

$$a.e. \quad Z\in B.$$

Recall the solution operator $\mathcal{Q}$ introduced in (4.4). Then, using Theorem 4.7, the following holds.

Corollary 4.8. *Let $\Omega^o\subseteq\Omega$ be Lipschitz domains as in* (4.1)-(4.2) *and recall the boundary decomposition $\partial\Omega = A\cup B$. Consider the boundary value problem*

$$\begin{cases} (\Delta+k^2)\vec{E} \;=\; \vec{0} \quad in\ \mathbb{R}^3\setminus\overline{\Omega},\\ \mathrm{div}\vec{E}=0 \quad in\ \mathbb{R}^3\setminus\overline{\Omega},\\ \vec{n}\times\vec{E}\Big|_{\partial\Omega}=\vec{g}\in L^2_{\tan}(\partial\Omega),\\ \vec{E}\ \textit{satisfies the radiation condition}\\ \qquad \textit{at infinity.}\end{cases} \tag{4.11}$$

Then $\mathcal{Q}(\vec{f})$ is the solution of (4.11) *where $\vec{f}\Big|_A := \vec{g}\Big|_A$, and $\vec{f}\Big|_B\in L^2_{\tan}(B)$ is a solution of the following boundary integral equation*

$$-\tfrac{1}{2}\vec{f}(Z)+\vec{n}(Z)\times \int_B\big[\blacktriangledown_Y\Gamma^o(Z,Y)\big]^t\vec{f}(Y)\,d\sigma(Y)=\vec{\mathcal{G}}(Z), \tag{4.12}$$

for

$$a.\ e.\ Z\in B,$$

where

$$\vec{\mathcal{G}}(Z) := \vec{g}(Z)-\vec{n}(Z)\times \int_A\big[\blacktriangledown_Y\Gamma^o(Z,Y)\big]^t\vec{g}(Y)\,d\sigma(Y). \tag{4.13}$$

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