OPTIMAL ELECTRODYNAMIC TETHER PHASING MANEUVERS

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Abstract
This paper studies the minimum-time orbit phasing maneuver problem for a constant-current electrodynamic tether (EDT). The EDT is assumed to be a point mass that is always in a plane perpendicular to the local magnetic field. After deriving and non-dimensionalizing the equations of motion, the only input parameters become current and the phase angle. Solution examples, including initial Lagrange costates, time of flight, thrust plots, and thrust angle profiles, are given for a wide range of current magnitudes and phase angles. The two-dimensional cases presented use a non-tilted magnetic dipole model, and the solutions are compared to existing literature. We are able to find similar trajectories to that of a constant thrust phasing maneuver, however, the time of flight is shorter. Full three-dimensional solutions, which use a titled magnetic dipole model, are also analyzed for orbits with small inclinations.

Nomenclature

- \( a \) Reference orbit radius
- \( B \) Magnetic Field Vector
- \( F \) Force vector
- \( F : \hat{l}_1, \hat{l}_2, \hat{l}_3 \) Magnetic Field Frame
- \( H \) Hamiltonian
- \( H_0 \) Total dipole strength
- \( L \) Tether direction vector
- \( m \) Mass
- \( \hat{m} \) Dipole direction unit vector
- \( n \) Reference mean orbit velocity
- \( N : \hat{n}_1, \hat{n}_2, \hat{n}_3 \) Inertial Coordinate Frame
- \( r \) Orbit radius
- \( R_o \) Radius of the earth
- \( \hat{R} \) Position unit vector
- \( T \) Dimensionless thrust
- \( T_0 \) Dimensionless thrust at original orbit radius
- \( TU \) Time Unit
- \( v_x, v_y, v_z \) Velocity components
- \( x, y, z \) Position components
- \( (\cdot) \) Inertial time derivative
- \( \phi \) Phase angle
- \( \lambda(\cdot) \) Lagrange costate
- \( \mu \) Gravity constant for attractive body
- \( \psi \) Thrust control angle
1 Introduction

Using an electrodynamic tether (EDT) as a means of propulsion is a relatively new concept that has not been fully developed. There is much in the literature concerning the trajectories of EDTs\(^1\)\(^–\)\(^6\), and Williams gives some solutions for optimal control of an EDT\(^1\)\(^–\)\(^2\). However, the equations of motion developed in the literature are based on dimensional parameters, making the results specific to the cases presented. Also the magnetic field model used in almost every case is a non-tilted dipole. Some other assumptions include restricting the EDT orbit to the equatorial plane and aligning the EDT with the nadir, or radial, direction. This is an unrealistic assumption for any practical application.\(^3\)\(^–\)\(^6\). All of these assumptions do not produce enough information create a usable control model.

2 Model

This section will describe the basic physics behind the EDT and how these basic principals lead to the development of the coordinate system used in analysis.

2.1 Physics

The EDT creates thrust by using a Lorentz force, which is defined by the following equation,

\[
F = iL \times B
\]  

where \(F\) is the force, \(i\) is the tether vector, \(L\) is the direction of the conductive tether, and \(B\) is the magnetic field vector. As the conductive tether moves through the earth’s magnetic field, the field induces a current through the wire to counter the change in magnetic flux passing through the tether. This creates a force in the opposite direction of flight, slowing the EDT and reducing its orbit altitude.\(^7\) Essentially the tether is converting orbit energy to electrical power in the tether. If the EDT supplies its own current, it can create a force in the direction of travel to accelerate the EDT and raise its altitude. Because the tether is not a circuit, it cannot complete the circuit on its own. It needs free electrons from the earth’s ionosphere. Devices on each end of the tether exchange electrons from the ambient plasma. An illustration of this process is seen in Figure 1. Because the EDT must operate in the ionosphere, the orbit altitude must be below about 2000 km. However, this range includes all manned missions and all other space applications requiring a low or medium earth orbit.

![Figure 1: Principals Behind EDT Operation\(^8\)](image)
2.2 Coordinate System and Magnetic Field Model

This optimization problem uses two coordinate frames: an Earth-centered initial frame, or ECI frame, and a magnetic field frame, F. The ECI frame is defined as $N : \hat{n}_1, \hat{n}_2, \hat{n}_3$, where $\hat{n}_1$ is in the Vernal Equinox direction, $\hat{n}_3$ is parallel to the earth’s axis of rotation, and $\hat{n}_2$ is the cross product of $\hat{n}_3$ and $\hat{n}_1$. The magnetic field frame, $F : \hat{l}_1, \hat{l}_2, \hat{l}_3$ is not as simply defined.

![Tilted Dipole Magnetic Field Model](image)

Figure 2: Tilted Dipole Magnetic Field Model

The $B$ vector points in the direction of the earth’s magnetic field at a given point in space. The magnetic field used in this problem is based on the tilted dipole model. In vector form, the following equation defines the direction and magnitude of the magnetic field

$$B(R) = \frac{R}{R^3} (3 (\hat{m} \cdot \hat{R}) \hat{R} - \hat{m}) \quad (2)$$

where $R$ is the radius of the earth, $H_0$ is the total dipole strength, $\hat{m}$ is the dipole direction, and $\hat{R}$ is the normalized position vector. The International Association of Geomagnetism and Aeronomy define the dipole direction and total magnetic field strength in its International Geomagnetic Reference Field report [9]. The unit vector of $B$ defines $\hat{l}_3$.

The two remaining unit vectors of the magnetic field frame, $\hat{l}_1$ and $\hat{l}_2$, require two reference planes to define them. Plane 1 contains both $\hat{n}_1$ and the dipole direction vector, $\hat{m}$. Plane 2 is perpendicular to $\hat{l}_3$. The line of nodes created by the intersection of these two planes defines $\hat{l}_1$, as depicted in Figure 3. The equation below shows the definition of $\hat{l}_1$ in more mathematical terms.

$$\hat{l}_1 = \hat{l}_3 \times (\hat{m} \times \hat{n}_1) \quad (3)$$

The cross product between $\hat{l}_3$ and $\hat{l}_1$ defines $\hat{l}_2$.

The thrust control angle, $\psi$, gives the direction of the force in the $\hat{l}_1, \hat{l}_2$ plane. Solving for this control angle at every time step is the main goal of this problem.
2.3 Equations of Motion

The derivation of the equations of motion follow the methods given by Thorne and Hall.\textsuperscript{10,11} First the second-order equations of motion are broken into first-order differential equations

\[
\begin{align*}
\dot{x} & = v_x \\
\dot{y} & = v_y \\
\dot{z} & = v_z \\
\dot{v}_x & = -\frac{\mu}{r^3}x + \frac{f_x}{m} \\
\dot{v}_y & = -\frac{\mu}{r^3}y + \frac{f_y}{m} \\
\dot{v}_z & = -\frac{\mu}{r^3}z + \frac{f_z}{m}
\end{align*}
\]

where \( r = \sqrt{x^2 + y^2 + z^2} \), \( f \) is a perturbation force, and \( \mu \) is the gravitational constant for the attracting body. All of these equations are with respect to the inertial frame. It is convenient to give the magnetic force vector in \( F \) frame coordinates. A coordinate transformation is required to rotate this vector to the inertial frame.

\[
N_f = [NP]^Pf
\]

\[
N_f = \begin{bmatrix}
C_{11} & C_{12} & C_{13} \\
C_{21} & C_{22} & C_{23} \\
C_{31} & C_{32} & C_{33}
\end{bmatrix} BiL \begin{bmatrix}
\cos \psi \\
\sin \psi \\
0
\end{bmatrix}
\]

\[
N_f = BiL \begin{bmatrix}
C_{11} \cos \psi + C_{12} \sin \psi \\
C_{21} \cos \psi + C_{22} \sin \psi \\
C_{31} \cos \psi + C_{32} \sin \psi
\end{bmatrix}
\]
where $C_{xx}$ is an element in the rotation matrix, $[NP]$. These force components are plugged into Equations 7, 8, and 9 to form

$$\dot{v}_x = -\frac{\mu}{r^3} x + \frac{BiL}{m} (C_{11} \cos \psi + C_{12} \sin \psi)$$  \hspace{1cm} (13)

$$\dot{v}_y = -\frac{\mu}{r^3} y + \frac{BiL}{m} (C_{21} \cos \psi + C_{22} \sin \psi)$$  \hspace{1cm} (14)

$$\dot{v}_z = -\frac{\mu}{r^3} z + \frac{BiL}{m} (C_{31} \cos \psi + C_{32} \sin \psi)$$  \hspace{1cm} (15)

Equations 4, 5, 13, 14, and 15 are then used to create the Hamiltonian, $H = \lambda \cdot \mathbf{d}$.

$$H = \lambda_x v_x + \lambda_y v_y + \lambda_z v_z + \lambda_x (\frac{\mu}{r^3} x + \frac{BiL}{m} (C_{11} \cos \psi + C_{12} \sin \psi)) + \lambda_y (\frac{\mu}{r^3} y + \frac{BiL}{m} (C_{21} \cos \psi + C_{22} \sin \psi)) + \lambda_z (\frac{\mu}{r^3} z + \frac{BiL}{m} (C_{31} \cos \psi + C_{32} \sin \psi))$$  \hspace{1cm} (16)

where $\lambda$ is the Lagrange costate for each position and velocity variable. The control law is then obtained by taking the partial derivative of $H$ with respect to $\psi$ and setting it equal to zero

$$\frac{\delta H}{\delta \psi} = \lambda_{xx} (-C_{11} \sin \psi + C_{12} \cos \psi) + \lambda_{xy} (-C_{21} \sin \psi + C_{22} \cos \psi) + \lambda_{xz} (-C_{31} \sin \psi + C_{32} \cos \psi) = 0$$  \hspace{1cm} (17)

$$\psi = \tan^{-1} \left( \frac{-\lambda_{xx} C_{12} - \lambda_{xy} C_{22} - \lambda_{xz} C_{32}}{-\lambda_{xx} C_{11} - \lambda_{xy} C_{21} - \lambda_{xz} C_{31}} \right)$$  \hspace{1cm} (18)

This control law is inserted into equations 13, 14, and 15 to get them in terms of the Lagrange costates.

$$\dot{\lambda}_x = \frac{\lambda_{xx} \mu}{r^3} - \frac{3 \mu x}{r^5} (\lambda_{xx} x + \lambda_{xy} y + \lambda_{xz} z)$$  \hspace{1cm} (22)

$$\dot{\lambda}_y = \frac{\lambda_{yy} \mu}{r^3} - \frac{3 \mu y}{r^5} (\lambda_{xx} x + \lambda_{xy} y + \lambda_{xz} z)$$  \hspace{1cm} (23)

$$\dot{\lambda}_z = \frac{\lambda_{zz} \mu}{r^3} - \frac{3 \mu z}{r^5} (\lambda_{xx} x + \lambda_{xy} y + \lambda_{xz} z)$$  \hspace{1cm} (24)

The other Lagrange costate differential equations are the negative partial derivatives of the Hamiltonian with respect to the spacial variable it is representing. The remaining equations of motion are as follows

$$\dot{\lambda}_{xx} = -\lambda_x$$  \hspace{1cm} (25)

$$\dot{\lambda}_{xy} = -\lambda_y$$  \hspace{1cm} (26)

$$\dot{\lambda}_{xz} = -\lambda_z$$  \hspace{1cm} (27)

This paper uses non-dimensional units in its analysis. To do this $\mu$ is set to unity and $BiL/m$ is represented by a dimensionless thrust variable, $T$. This done by using the following expression,

Bitzer, Matthew
\[ T = \frac{BiL/m}{man^2} \]  

(28)

where \( a \) is reference orbit distance and \( n \) is a reference mean orbit velocity.

The twelve optimal equations of motion assume the EDT and the attracting body are a point masses and that the only other external force is the Lorentz force. They are constructed in such way that they describe minimum-time motion. In other words, the spacecraft at any position reached that position in minimum time. This makes time of flight a cost function in the optimization problem. The other cost function used is the residual between the position and velocity of the spacecraft at \( t = t_f \) and the target position and velocity at \( t = t_f \). The EDT meets its target when the residual approaches zero.

2.4 Algorithm

The first step of the algorithm uses simulated annealing (SA). It mimics the physical process of annealing which takes a highly unorganized, melted material and reduces the temperature slowly so that the crystalline structure is nearly perfect. SA involves lowering the energy state, or performance index, of the system slowly so that its probability functions can find possible minima. As the energy state lowers, the amount of possible minima decreases until only the global minimum remains.

Another way of looking at the SA process is by using the topography of the earth’s surface as an analogy. The objective is to find the lowest point on Earth. The algorithm would start at a “temperature” of 30,000 feet. At this height, all of the earth is a local minimum. As the altitude is reduced, certain regions, such as the Rocky Mountains or the Himalayas, can no longer be a minimum. As the height reduces even further, the only possible minima remaining are Death Valley and the Dead Sea. If the altitude is lowered enough, only one possible region is left, the Dead Sea.

The problem with SA is that it has a low convergence rate and low accuracy. The accuracy is refined by using an optimizer, such as fsolve in MATLAB. Optimizers have very small convergence radii but high convergence rates and high accuracy. The last step uses the solution of one optimization problem to obtain the solution to a similar problem using continuation.

To obtain full three-dimensional optimal control solutions, comparable two-dimensional solutions must be found first. In the two-dimensional problem, the target orbit is in the earth equatorial plane and \( \hat{m} \) points in the \( -\hat{n}_3 \) direction. This, in turn, keeps the electromagnetic force in the \( \hat{n}_1 - \hat{n}_2 \) plane. Also in the two-dimensional case, the \( N \) frame and the \( F \) frame become one of the same.

The SA algorithm finds an initial guess for a two-dimensional system which is then refined to a solution within a specified tolerance. The two-dimensional solution is used as an initial guess for a full tilted dipole problem in three-dimensional space.

3 Results

3.1 Two-Dimensional Optimal Control Problem

Several conclusions can be drawn from studying the two-dimensional optimal problem. As a reference, the following results are compared to the work of Hall and Collozo-Perez. Their work focused on constant thrust, minimum-time optimal phasing maneuvers in two-dimensional space. The authors provided several solutions that were representative of their work and trends that encompassed their results. Two aspects of their work can be directly compared to the two-dimensional EDT optimal control problem. One comparison is obviously the trajectory. The other is the near-invariance of the initial costates when changing the dimensionless thrust and phase angle by an order of magnitude.

3.1.1 Trajectory Analysis

When comparing the optimal trajectory of a constant current EDT and a constant thrust spacecraft, intuition suggests that the EDT will catch up to its target faster because thrust increases when the orbit radius decreases in accordance with Equation 1. A higher thrust should propel the EDT to the target more quickly than a constant thrust model.
Calculations show that the EDT does indeed arrive at its target more quickly. Figures 5 and 6 show the trajectories for the EDT and a spacecraft with constant thrust. The red, dotted line is the original orbit and the blue, solid line is the trajectory. In the case depicted, $T_0 = 0.5$ and $\phi = 1.46$ rad, where $T_0$ is the dimensionless thrust at the original orbit radius. This is an unusually large thrust value for an EDT, but this example was chosen so that differences in their trajectories would be obvious on a plot. The plot shows the EDT arriving at its target about 0.3 TU before the constant thrust spacecraft arrives. Also the shape of the path mid-flight is more rounded for the constant thrust trajectory than for the EDT trajectory. This shows that the thrust increased mid-flight because the EDT was able to almost fly straight toward its target while the constant thrust spacecraft had to follow a more rounded course.

![Figure 5: EDT Trajectory](image1)

![Figure 6: Constant Thrust Trajectory](image2)

As stated before, a large phase angle and a large thrust were used to illustrate the EDT’s shorter time of flight, but this trend is apparent in all cases. When comparing initial costates presented in Hall and Collozo-Perez with the EDT initial costates for several cases, the EDT reached it’s target faster every time. Table 2 shows this fact. The difference in time of flight is more pronounced in large phase angle maneuvers. For the case with $\phi = 0.54$, the difference in time of flight is 0.0882 Time Units (TU) while the difference in time of flight is only 0.001525 TU for $\phi = 0.0074$ rad. The small phase angle does not allow the EDT to dip far enough into the orbit to use its advantage over the constant thrust spacecraft.

<table>
<thead>
<tr>
<th></th>
<th>$T_0 = 0.5$, $\phi = 0.42$</th>
<th>$T_0 = 0.5$, $\phi = 0.54$</th>
<th>$T_0 = 0.05$, $\phi = 0.89$</th>
<th>$T_0 = 0.005$, $\phi = 0.0074$</th>
<th>$T_0 = 0.005$, $T = 0.005$</th>
</tr>
</thead>
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<tr>
<td>EDT</td>
<td>$\lambda_{y0}$</td>
<td>0.42896</td>
<td>0.099345</td>
<td>0.34992</td>
<td>0.18569</td>
</tr>
<tr>
<td></td>
<td>$\lambda_{x0}$</td>
<td>0.62643</td>
<td>0.65774</td>
<td>0.44331</td>
<td>0.72805</td>
</tr>
<tr>
<td></td>
<td>$\lambda_{y+0}$</td>
<td>0.42613</td>
<td>-0.039660</td>
<td>0.98215</td>
<td>0.000697</td>
</tr>
<tr>
<td></td>
<td>$t_f$</td>
<td>2.51007</td>
<td>1.95232</td>
<td>5.42443</td>
<td>2.45092</td>
</tr>
<tr>
<td>Constant Thrust</td>
<td>$\lambda_{y0}$</td>
<td>0.47313</td>
<td>0.14480</td>
<td>0.34156</td>
<td>0.18627</td>
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<tr>
<td></td>
<td>$\lambda_{x0}$</td>
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<td>0.43092</td>
<td>0.72812</td>
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<tr>
<td></td>
<td>$\lambda_{y+0}$</td>
<td>0.49960</td>
<td>0.00012</td>
<td>1.00027</td>
<td>0.001362</td>
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<td></td>
<td>$t_f$</td>
<td>2.78398</td>
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<td>5.58198</td>
<td>2.45245</td>
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Figures 7 and 8 show the control angle, $\psi$, for each spacecraft. For the two-dimensional case, $\psi$ is measured with respect to $\hat{n}_1$. The shapes are very similar with each control angle starting in the third quadrant to lower the orbit and increase speed. The thrust then spins around at the half way point to raise the orbit to meet with the target. This is true for all cases studied.
3.1.2 Near-Invariance of the Initial Lagrange Costates

One important discovery in Hall and Collozo-Perez is that when $T$ and $\phi$ are both increased by a order of magnitude, the values of the initial Lagrange costates and the time of flight do not change by more than 0.10 for moderate and low thrust cases. Having near-invariance of the initial costates is important because it allows for the calculation of a whole family of solutions once one solution is found. This trait is also present in the EDT two-dimensional optimal control problem.

Table 3 gives three representative examples that validate the claim of near-invariance for moderate and low thrust cases and shows its limitations. The first example has the lowest thrust in the group, but the initial costates show the least amount of variance with the two sets identical to three significant figures. The second example shows an invariance between the two costate sets with a precision of two significant digits. The last example compares a high thrust, $T = 0.5$, with a thrust one order of magnitude smaller. This comparison shows the most variation of all the three cases. Further exploration showed large variations in the initial costates when a large thrust and phase angle is dropped by an order of magnitude. The large variations in the costates are caused by the different ranges of thrust magnitudes experienced during the maneuver. There is a larger difference in the thrust range experienced during a maneuver by a $T = 0.5$ to $T = 0.05$ thrust drop than there is for a $T = 0.005$ to $T = 0.0005$ thrust drop. This is the reason for the greater variance between costate sets as thrust increases.

<table>
<thead>
<tr>
<th>$T$</th>
<th>$\phi$</th>
<th>$\lambda_{\nu_0}$</th>
<th>$\lambda_{\nu_{\phi}}$</th>
<th>$\lambda_{\nu_{\psi}}$</th>
<th>$t_f$</th>
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<tr>
<td>1</td>
<td>0.0005</td>
<td>0.00074</td>
<td>0.18527</td>
<td>0.72864</td>
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</tr>
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<td>0.0074</td>
<td>0.18569</td>
<td>0.72805</td>
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<tr>
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</tr>
<tr>
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<td>0.34688</td>
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<td>0.66314</td>
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<tr>
<td>3</td>
<td>0.05</td>
<td>0.146</td>
<td>0.38861</td>
<td>0.71940</td>
<td>0.32728</td>
</tr>
<tr>
<td>0.5</td>
<td>1.46</td>
<td>0.42896</td>
<td>0.62643</td>
<td>0.42613</td>
<td>2.51007</td>
</tr>
</tbody>
</table>

3.2 Three-Dimensional Optimal Control Problem

Now that two-dimensional solutions for the EDT optimal control problem have been established, they can be used as initial guesses to full three-dimensional solutions using the tilted-dipole model. These initial guesses proved to only be useful to find zero inclination phasing maneuvers. Also not all two-dimensional solutions could be used as initial guesses. Solutions with high phase angles or high thrust did not converge when using fsolve. When a three-dimensional, zero inclination angle solution was found, the reason behind the small convergence radius became clear. The F frame varies greatly with time relative to the inertial frame. This is due changes in the $B$, and by extension the $l_3$ vector. As of the writing of this paper, the most
inclined phased orbit maneuver that has converged is 5 degrees. Figure 9 shows a time history of the $F$ frame orientation with respect to the $N$ frame in ten equal increments over a period of about 3 TU. Because the thrust vector is defined to be only in the $\hat{1} - \hat{2}$ plane, the solver has trouble converging to an answer. Once a solution is found, one must consider the dynamics of the solution. Since the thrust is always perpendicular to the magnetic field vector, the tether must be as well for maximum efficiency. For this optimal control to work, the tether must not only rotate about $\hat{3}$, but it must also rotate as the $F$ rotates.

![Figure 9: Time History of $F$ Frame Rotation. Red = $\hat{1}$, Green = $\hat{2}$, Blue = $\hat{3}$, Black = $N$ frame](image)

### 4 Conclusion

Solutions to the minimum-time, constant-current orbit phasing maneuver for an electrodynamic tether have been found for a wide range of two-dimensional cases as well as limited three-dimensional cases. The two-dimensional results were compared with existing data from the literature to provide insight into the characteristics of the solutions. In all cases, the EDT reached its target faster than a constant thrust spacecraft because the Lorentz force acting on the tether increases as orbit radius decreases. Also sets of initial Lagrange costates, whose parameters $T$ and $\phi$ each vary by an order of magnitude, were shown to be near-invariant for moderate to low values of thrust and phase angle. These two-dimensional solutions were used as initial conditions to solve for three-dimensional solutions which included a tilted-dipole magnetic field model. These initial conditions were only useful for zero inclination orbits. The most inclined orbit phasing maneuver solved at the time of publication is at 5 degrees. The reason for the difficulty in converging to a more inclined solution is due to the rotation of the $F$ frame.

Further research is needed to fully develop the three-dimensional solutions for wider range of inclination angles. The behavior of the $F$ frame becomes more erratic as inclination angle increases and solution methods must be found to get around this. After this is completed, the same method described in this paper can be used to find optimal trajectories for other simple maneuvers such as inclination angle change and orbit radius change. These basics must be developed in order for this technology to become practical.

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References


