

SYSTEM IDENTIFICATION METHODS BASED ON THE PROPER ORTHOGONAL DECOMPOSITION

Timothy C. Allison
Virginia Polytechnic Institute and State University, Blacksburg, Virginia, 24061

Abstract

The proper orthogonal decomposition is a powerful engineering analysis tool that has historically been used to reduce the dimension of large finite element models. The usefulness of this approach is limited because it requires the construction of a full-order finite element model. This research outlines a method for using experimental data to construct a system model that can be used to simulate a system's response to various initial conditions without requiring a finite element model. This modeling approach is particularly powerful because it may even be applied to a nonlinear system to obtain the optimal linear model for the system. Sample models were created using this method for both linear and nonlinear beams and the responses to various initial conditions are compared with the response generated by full-order finite element models.

Introduction

The Finite Element (FE) method is commonly used to analyze the dynamics of complex structures¹. Although the method is very powerful, some of its limitations become apparent when it is applied for analysis of very large, possibly nonlinear structures with millions of degrees of freedom. Months or years are often required to develop the geometry and form the element mesh for such models. After the model is completed the analysis may require days or weeks of processing time². Finally, there is no guarantee that the analysis will accurately predict behavior of an actual structure. The analysis may be incorrect due to modeling errors (e.g., incorrect assumptions about damping or linearity), parameter errors (e.g., inaccuracy of Young's modulus), or other factors³.

Many methods have been developed to circumvent these problems by using experimental response data to characterize or identify a system. Modal analysis is commonly used to validate or update linear FE models to insure that the computer analyses will be correct⁴. Other identification methods have been developed for both linear and nonlinear systems^{5, 6}. However,

current nonlinear identification techniques are only developed for systems with a small class of nonlinearities and nonlinear system identification is currently an active field of research³.

This paper describes a method for using the Proper Orthogonal Decomposition (POD) as a method for using response data to construct a predictive model for linear and nonlinear systems. The POD is a statistical method for extracting significant shapes and their corresponding amplitude modulations that are present in a displacement field history⁷. The POD is an attractive tool because it is a linear procedure and its governing mathematics is therefore relatively straightforward. However, it is often applied to nonlinear problems to compute the "optimal approximating *linear* manifold for the data"⁷. Although using a linear manifold to approximate a nonlinear system necessarily introduces error, the POD does "not do the physical violence of linearization methods"⁸. The POD, then, offers an alternate and more accurate route for using linear methods to analyze nonlinear systems than linearizing the governing equations of motion.

This paper is organized into four sections. The first section describes how the POD is calculated from an experimental response using the singular value decomposition and explains the significance of each element of the POD. The second section describes methods for using the POD to predict the free response of a structure to initial displacements or velocities. The third section applies these methods to two example problems: a linear cantilever beam and a jointed, damped beam. The predicted responses are compared with those obtained from finite element models for each beam. The final section discusses the accuracy of the POD-based response prediction methods.

Computation of the POD

The POD can be computed by many methods⁷. This section explains how the POD is computed with the singular value decomposition of a snapshot matrix. First, a system response is generated by either forcing the system or imposing an initial condition. In this paper, we will assume

that an initial displacement profile \mathbf{w}_0 is used to generate a response. Next, the displacement at m degrees of freedom is sampled n times and the data are arranged in a “snapshot” matrix W :

$$W = \begin{bmatrix} w_1(t_1) & w_1(t_2) & \cdots & w_1(t_n) \\ w_2(t_1) & w_2(t_2) & & \\ \vdots & & \ddots & \\ w_m(t_1) & & & w_m(t_n) \end{bmatrix} \quad (1)$$

Next, the singular value decomposition of W is computed:

$$W = U\Sigma V^T \quad (2)$$

In Eq. (2), the columns \mathbf{u}_i of U are the POMs, the columns \mathbf{v}_i of V are the proper orthogonal coordinate (POC) histories that correspond to each POM, and Σ is a diagonal matrix whose diagonal elements σ_i are the proper orthogonal values (POVs) corresponding to each POM. The POC histories describe the amplitude modulation of each POM and the POVs describe the relative significance of each POM in W^T . If special conditions are met, the POMs will be equal to the linear normal modes (linear systems) or linear approximations to nonlinear normal modes (nonlinear systems)^{9, 10}. The percentage of signal energy captured by \mathbf{u}_i is given by

$$\varepsilon_i = \frac{\sigma_i}{\sum_{j=1}^m \sigma_j} \quad (3)$$

Typically, only POMs that constitute a certain percentage of signal energy (e.g. 99% or 99.9%) are considered^{7, 8}. If k dominant POMs are considered then we may approximate W as a summation of POMs and corresponding POC histories, shown below (noting that Σ is diagonal):

$$W \approx \sum_{i=1}^k \sigma_i \mathbf{u}_i \mathbf{v}_i^T \quad (4)$$

We note that even signals generated by nonlinear systems may be represented by a summation of

POMs³. It should be noted that the POMs and POC histories are orthonormal:

$$U^T U = V^T V = I \quad (5)$$

Although this paper has focused on calculation of the POMs, POVs, and POC histories by performing a singular value decomposition, these quantities may also be determined when calculating the POD by other methods⁷.

Most applications of the POD use only the POMs, meaning that only the spatial information about the system is obtained. The only documented use for the POC histories is to examine their frequency content to determine which linear normal modes are represented in each POM⁷. This paper describes a new application for the POC histories by using them to form a predictive model for the free response a system.

Free Response Prediction

This section provides a method for using the POMs and POC histories obtained from a response to predict the response to other initial conditions. As shown in Eq. (4), the matrices U , V and Σ describe completely a system's response to \mathbf{w}_0 without requiring information about the full-order equations of motion. We now wish to modify the matrices to describe the system's response to a different initial displacement profile $\tilde{\mathbf{w}}_0$. When solving vibration problems (e.g. the wave equation) analytically using separation of variables, a typical approach is to express the response as a summation of spatial eigenfunctions, temporal functions, and coefficients that indicate the relative significance of each mode in the response. While the eigenfunctions and temporal functions do not depend on the initial conditions, the significance coefficients do depend on them and are calculated using inner products of the eigenfunctions with the initial displacement profile¹¹. We will take a similar approach and assume that U and V do not change for a given system, but that the participation of each POM, measured by σ_i , changes to represent the response to new initial conditions. If these assumptions are made then the response \tilde{W} to $\tilde{\mathbf{w}}_0$ may be written as

$$\tilde{W} \approx \sum_{i=1}^k \tilde{\sigma}_i \mathbf{u}_i \mathbf{v}_i^T \quad (6)$$

where the tilde notation indicates that the values in the diagonal matrix $\tilde{\Sigma}$ have changed, although $\tilde{\Sigma}$ is still diagonal. The first column of Eq. (6), which corresponds to the initial time $t = t_0$, is

$$\tilde{\mathbf{w}}_0 \approx \sum_{i=1}^k \tilde{\sigma}_i \mathbf{u}_i v_{i,0} \quad (7)$$

In Eq. (7), the scalar $v_{i,0}$ is the first element in each POC history \mathbf{v}_i . Multiplying both sides of Eq. (7) on the left by \mathbf{u}_j^T and recalling the orthonormality of the POMs we may solve for $\tilde{\sigma}_j$:

$$\tilde{\sigma}_j = \frac{\mathbf{u}_j^T \tilde{\mathbf{w}}_0}{v_{j,0}}, j = 1, 2, \dots, k. \quad (8)$$

We note that $v_{j,0}$ in Eq. (8) is nonzero because the original POVs are found from

$$\sigma_j = \frac{\mathbf{u}_j^T \mathbf{w}_0}{v_{j,0}}, j = 1, 2, \dots, k. \quad (9)$$

and are finite. After the new POVs $\tilde{\sigma}_j$ have been calculated, the response to $\tilde{\mathbf{w}}_0$ may be approximated using Eq. (6).

By using the new POVs calculated from Eq. (8) (10) in Eq. (6), we are able to simulate the free response of a system to a variety of initial conditions using only data obtained from the original POD. This method will most accurately predict the structural response to initial conditions that generate shapes in the response similar to the POMs \mathbf{u}_i . It is desirable to choose initial conditions that excite the widest variety of shapes in the response in order to generate POMs that can represent responses to a large selection of initial conditions.

Examples

This section applies the methods described in the previous section to two models: an undamped beam and a jointed beam with a dashpot at the end, shown in Fig. 1. Finite element models were created for each beam, and a four-parameter Iwan model¹² was used to model the joint as a nonlinear vertical spring. First, these models were used to

simulate the exact response of each beam to several initial displacement profiles, shown in Fig. 2. Each displacement profile was generated by applying static loads to the beam. The free responses of each model to the initial displacement profile \mathbf{w}_0 were simulated for 0.05 seconds and the vertical displacements at 24 points (linear

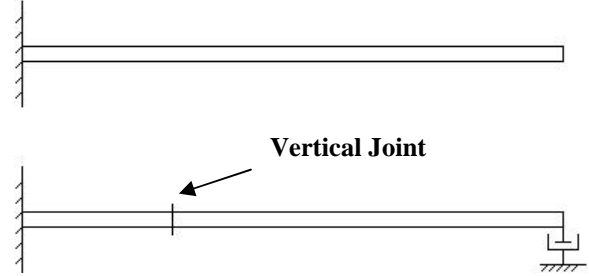


Fig. 1 Linear (top) and Nonlinear (bottom) Beam Models

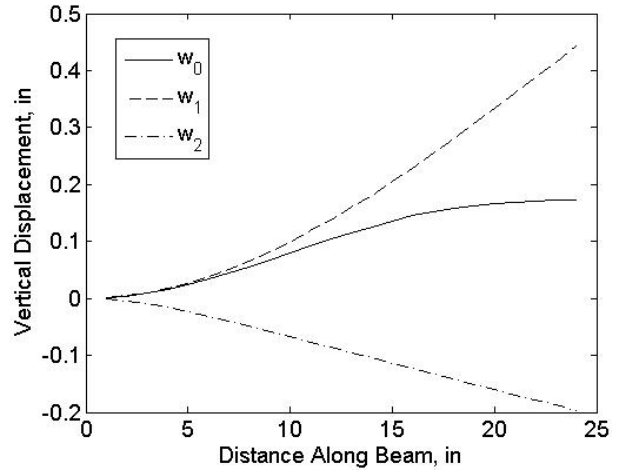


Fig. 2 Applied Initial Displacement Profiles

beam) and 25 points (nonlinear beam) were captured every 0.1 milliseconds to form \mathbf{W} . Thus the dimensions of \mathbf{W} were (24 x 500) for the linear model and (25 x 500) for the nonlinear model. The nonlinear beam displacement was captured next to both sides of the joint to allow for representation of the joint slip.

Next, the POD was computed for each beam, and the methods described in section Eqns. (6) - (9) were applied to simulate the responses of both systems to initial displacement profiles \mathbf{w}_1 and \mathbf{w}_2 . The responses were simulated using the first four POMs, which corresponded to 99.9% of the signal energy. The original POVs (from the

response to w_0) and recalculated POVs (for responses to w_1 and w_2) are shown in Fig. 3 for the linear beam.

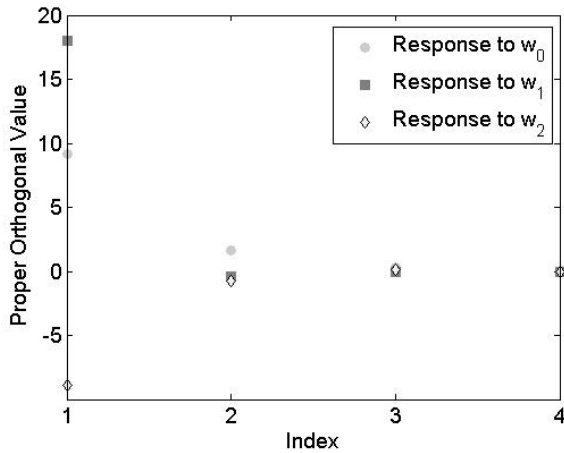


Fig. 3 Original and Recalculated POVs for Linear Beam Model

The original and recalculated POVs for the nonlinear beam were very similar. In both cases, the POVs drop off very quickly, showing that only a few POMs are necessary to represent the motion accurately. This figure also illustrates that the significance of each mode changes in response to the different initial displacement profiles. For example, the first POV for both beams is larger in response to w_1 than to w_0 , indicating that the first POM is more active in the response to w_1 than to w_0 . The first three POMs for each beam are shown in Figs. 4 and 5. The nonlinearity at the joint is represented by a vertical discontinuity at the joint location (8 inches from the root of the beam) in each of the POMs for the nonlinear beam.

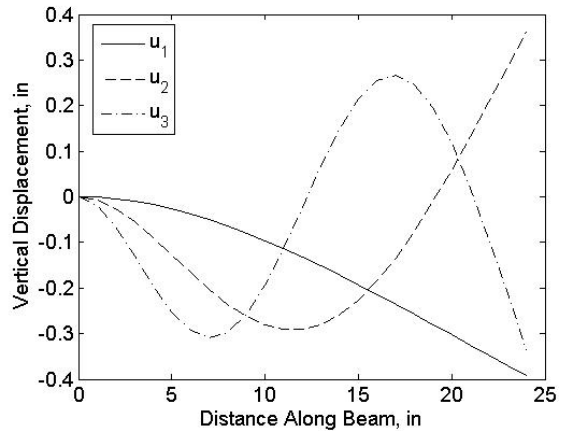


Fig. 4. First Three POMs for Linear Beam Model

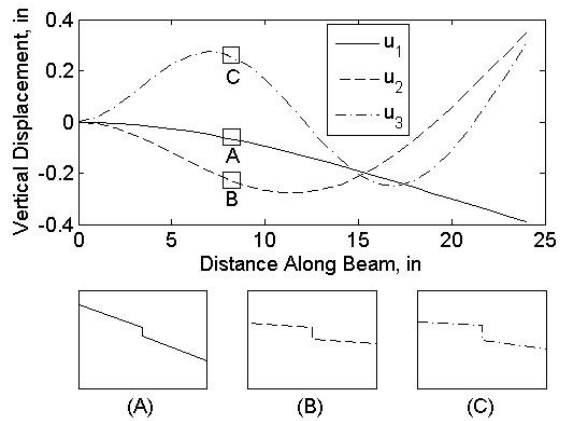


Fig. 5 First Three POMs for Nonlinear Beam Model

The tip displacements at each time step calculated by the FE model and the POD-based model for the linear beam responses are shown in Figs. 6-7. Both figures show that the POD-based model predicts the tip displacement very accurately for the linear beam. The tip displacements for the nonlinear beam responses are shown in Figs. 8-9, and the joint slip in the nonlinear model for both responses is shown in Figs. 10-11. In each figure the results obtained using the POD-based models are plotted with the exact results obtained by the FE models. The amplitude

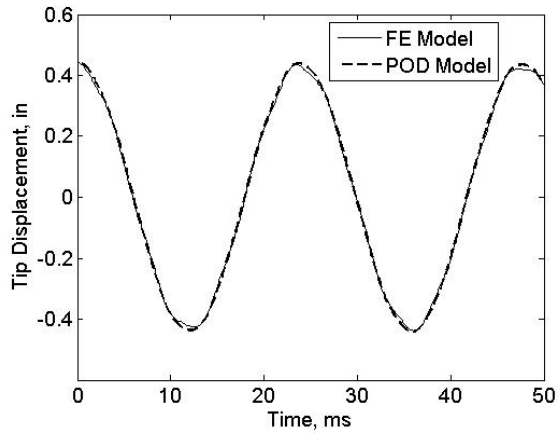


Fig 6. Tip Displacement of Linear Beam Model in Response to w_1

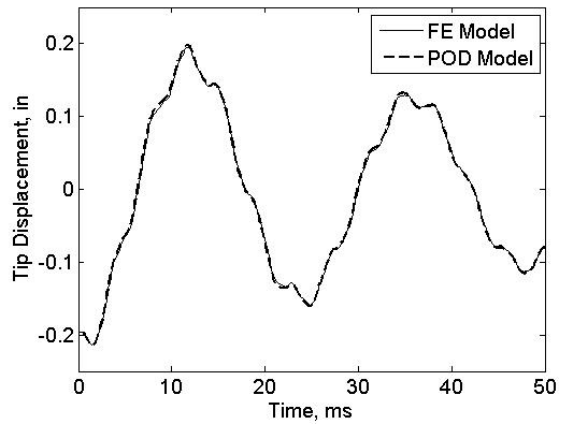


Fig 9 Tip Displacement of Nonlinear Beam Model in Response to w_2

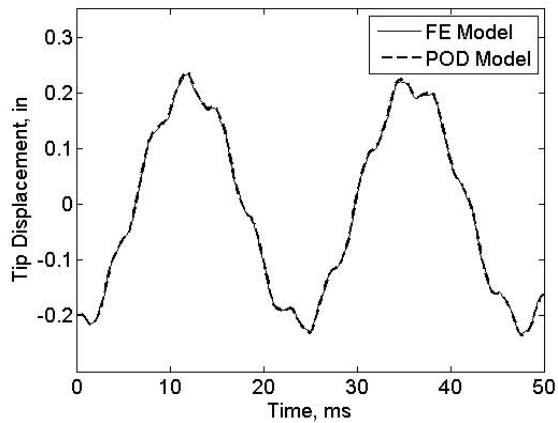


Fig. 7 Tip Displacement of Linear Beam Model in Response to w_2

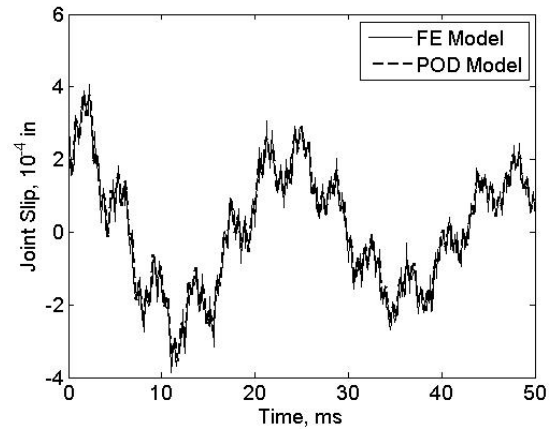


Fig. 10 Joint Slip in Nonlinear Beam Model in Response to w_1

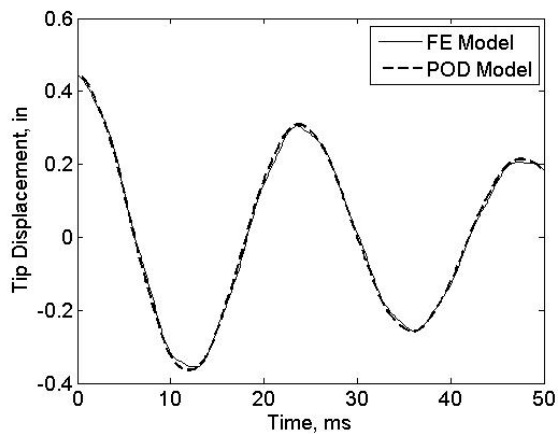


Fig. 8 Tip Displacement of Nonlinear Beam Model in Response to w_1

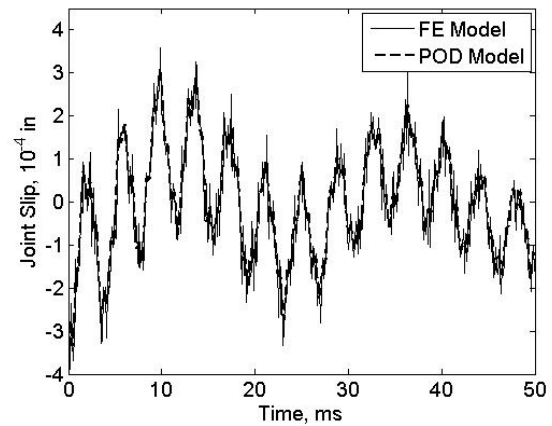


Fig. 11 Joint Slip in Nonlinear Beam Model in Response to w_2

of the tip displacement and joint slip decreases with time due to the presence of the dashpot. The joint slip is calculated from the difference in the vertical displacement of the beam on either side of the joint.

Conclusions

The results displayed in the previous section demonstrate excellent agreement between the finite element models and the reduced-order POD models using only four POMs for the linear beam and nine POMs for the nonlinear beam. The POD models formulated from the response to the initial displacement profile \mathbf{w}_0 was able to predict the response to new displacement profiles for both the linear and nonlinear beam accurately. The POD model of the nonlinear beam successfully captured the energy dissipation resulting from the dashpot as well as the nonlinear joint slip. Although the example analyses shown were performed with finite element models, the results suggest that an accurate POD model for a structure may be obtained by time sampling the transient response of a structure in an experiment.

This paper outlined a method for using the POD to construct a predictive model for the free response of a structure using mode summation techniques. The POMs and corresponding POC histories obtained by imposing an initial displacement profile may be applied to predict the response to other displacement profiles by recalculating the POVs associated with each POM. Both the POMs and the POC histories are easily obtained by computing the singular value decomposition of a matrix of displacement field snapshots. No knowledge of the governing equations of motion for the structure (e.g., from a finite element model) is required to construct the POD model. Thus, the proposed method may provide a useful technique for modeling complex systems using only test data.

Acknowledgments

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