AEROELASTIC ANALYSIS FOR NEAR-SPACE APPLICATIONS IN ULTRA-LARGE ASPECT RATIO WINGS

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Abstract

This research describes an aeroelastic modeling method for use within an optimization framework combining rigid and inflatable wing design. The inflatable portion of the wing is stowed within the rigid portion of the wing when not in use; this is done using a telescoping spar system. The aeroelastic analysis of the wing is critical in determining a feasible design using a multidisciplinary design optimization framework, and the approach must handle the discontinuities between sections of the wing. The natural modes are determined using the Rayleigh-Ritz method, while continuity between each section of the wing is handled using a penalty approach. The penalty parameters are determined and the method is verified using a comparison of the predictions from this method with known natural modes for simple examples.

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Introduction

Ultra-large aspect ratio aircraft require different design modeling from those with moderate aspect ratios. Ultra-large aspect ratio aircraft are generally more flexible structures than the traditional wing; therefore, they require modeling methods capable of handling a flexible structure even at the preliminary design stage. The design suggested here also investigates the potential for a hybrid between rigid and inflatable wings. Inflatable wings provide benefits in wing structural weight and can be packed for easier and more cost effective storage and transportation [1] and thus can be easily deployed. The inflatable section can be deployed via a telescoping spar and mechanisms. A multi-stepped beam structural model is used to analyze the wing structure. The multi-stepped beam method has been used by many [2, 3, 4, 5, 6] to model the structural nature of more flexible wings, or wings with multiple sections; however, there are many different methods for handling the structural analysis, specifically the discontinuities, in the stepped beam.

While many analytical and numerical investigations of stepped beams have been conducted, most have been for beams with a single discontinuity or centrally-stepped
Euler-Bernoulli beams. Yavari, Sarkani, and Reddy [2] analyzed non-uniform Euler-Bernoulli and Timoshenko beam with discontinuities, and Biondi and Caddemi [3] derived solutions of Euler-Bernoulli beams with singularities; both used generalized functions to handle discontinuities. Lu et al’s [4] composite element method for multi-stepped beam analysis has the advantage of being able to treat the whole beam as a uniform beam, but the methods are much more complex than a traditional finite element method. Jaworski and Dowell [5] successfully investigated multi-stepped beams using a component modal analysis and Lagrange multipliers to handle the discontinuities; however, using Lagrange multipliers adds much complexity to modeling, as well. Dang, Kapania, and Patil [6] employed a Ritz approach with local trigonometric functions and a Lagrange multiplier approach to handle discontinuities. Kapania and Liu [7] used the Ritz method with trial functions and penalties to handle discontinuities in equivalent-plate models and showed a validation of this method compared with MSC/NASTRAN. Slemp, Kapania, and Mulani [8] investigate solving static boundary value problems using the integrated local Petrov-Galerkin sinc method; boundary conditions are handled using both the traditional penalty approach and Lagrange multipliers. Penalty approaches can be used to handle discontinuities in a simpler manner than is provided by using Lagrange multipliers.

This work details the best method from those described above for use in multidisciplinary design optimization. The model must be able to handle the discontinuity between rigid and inflatable sections (i.e. the change in cross-section and material properties). Natural modes for this model are determined using the Rayleigh-Ritz method; discontinuities in the structure have been accounted for in two different ways using this method. The Lagrange multiplier [5, 6] and penalty [7, 8] approaches are among the most used. Both methods enforce the geometric constraints at the discontinuity. Similarly, both offer accuracy through analytic computation. The Lagrange multiplier approach increases the number of equations since the Lagrange multipliers must be determined first in order to arrive at the natural modes. Solving for Lagrange multipliers requires a dynamic, graphically based model, and it is not amenable to an automatic determination of natural modes. The penalty approach is more robust, provided the penalty parameter is carefully chosen. Gern, Inman, and Kapania [9, 10] were able to apply this method to morphing wing models. The penalty approach will be used in this work since it provides a better modeling solution and has previous validation for use in this type of work.

Aeroelastic Analysis

The natural modes of a multi-stepped cantilever beam are derived based on the Rayleigh-Ritz method. The penalty approach is an accurate method for handling discontinuities. This method provides automatic determination of modes, which is critical for its application to the design optimization.

Structural Model

The equations of equilibrium for the wing can be derived using Lagrange’s equations

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\xi}_i} \right) - \frac{\partial L}{\partial \xi_i} = \Xi_i, \ i = 1, 2, ..., n \quad (1)$$

where $\xi_i$ is the generalized coordinate $i$, $\Xi_i$ is the generalized nonconservative forces, and
where \( T \) is the kinetic energy and \( U \) is the strain energy. The \( () \) is the derivative with respect to time. The strain energy of the multi-stepped beam is derived by decomposing the beam into \( c \) components of constant cross-section (or material properties). The penalty approach enforces continuity between components using springs with large stiffness. The total strain energy of the beam becomes:

\[
U = U_{\text{beam}} + U_{\text{springs}},
\]

where \( U_{\text{beam}} \) is composed of strain energy due to bending, \( w^{(c)} \), and torsion, \( \bar{\theta}^{(c)} \) and \( U_{\text{springs}} \) is the strain energy in the springs.

\[
U_{\text{beam}} = \frac{1}{2} \sum_{c=1}^{N} \int_{0}^{l} \left[ EI^{(c)} w'^{(c)^2} + GJ^{(c)} \bar{\theta}'^{(c)^2} \right] dy
\]

\[
U_{\text{springs}} = \frac{1}{2} \sum_{c=1}^{N-1} k_{w}^{(c)} \left( w^{(c)}(L^{(c)}) - w^{(c+1)}(0) \right)^2
+ \frac{1}{2} \sum_{c=1}^{N-1} k_{\theta}^{(c)} \left( \bar{\theta}(L^{(c)}) - \bar{\theta}(c+1)(0) \right)^2
\]

The bending stiffness \( EI^{(c)} \) and effective torsional stiffness \( GJ^{(c)} \) are constant for each section of the beam. The \( ()' \) is derivative with respect to \( y \). The kinetic energy is:

\[
T = \frac{1}{2} \sum_{c=1}^{N} \int_{0}^{l} \left[ m^{(c)} \dot{w}^{(c)^2} + 2m^{(c)} d^{(c)} w^{(c)} \dot{\bar{\theta}}^{(c)} + J_{o}^{(c)} \dot{\bar{\theta}}'^{(c)^2} \right] dy
\]

where \( d^{(c)} \) is the distance between the shear center and the center of gravity (for a symmetric airfoil, \( d = 0 \)).

Using the Rayleigh-Ritz method the response is approximated using

\[
w^{(c)} = \sum_{i=1}^{N_{w}^{(c)}} \eta_{i}^{(c)} \Psi_{i}^{(c)}(\bar{y})
\]

\[
\bar{\theta}^{(c)} = \sum_{i=1}^{N_{\theta}^{(c)}} \phi_{i}^{(c)} \Theta_{i}^{(c)}(\bar{y}),
\]

where \( \bar{y} = y/L^{(c)} \) and

\[
\Psi_{i}^{(1)} = \bar{y}^{i+1}, \quad i = 1, 2, ..., N_{w}^{(1)},
\]

\[
\Psi_{i}^{(c)} = \bar{y}^{i-1}, \quad i = 1, 2, ..., N_{w}^{(c)},
\]

\[
c = 2, 3, ..., N
\]

The first equation, Eq. 11, enforces cantilever beam boundary conditions, whereas the second equation, Eq. 12, enforces free-free boundary conditions for the remaining components.

\[
w^{(1)}(0) = 0, \quad w^{(1)}(1) = 0
\]

\[
w^{(c)}(0) \neq 0, \quad w^{(c)}(1) \neq 0
\]

The natural beam modes are computed by substituting the approximate functions in the Lagrange equations with the generalized forces \( \Xi_{i} = 0 \). The stiffness matrix is computed from the strain energy term and the mass matrix is computed from the kinetic energy term. To simplify the notation consider only a beam with three components and neglect torsion modes for now. The stiffness matrix due to beam bending only is as shown in Eq. 13 where \( i, j = 1, 2, ..., N_{w} \) are the number of terms included in the series. Here we use same number of terms for all components. The size of the stiffness matrix
springs in bending is shown in Eq. 15.

\[
K_b = \begin{bmatrix}
K_b^{(1,1)} & 0 & 0 \\
0 & K_b^{(2,2)} & 0 \\
0 & 0 & K_b^{(3,3)}
\end{bmatrix}
\]  \hspace{1cm} (13)

The stiffness matrix of the highly stiff springs in bending is shown in Eq. 15.

\[
K_{b_s} = \begin{bmatrix}
K_{b_s}^{(1,1)} & K_{b_s}^{(1,2)} & 0 \\
K_{b_s}^{(2,1)} & K_{b_s}^{(2,2)} & K_{b_s}^{(2,3)} \\
0 & K_{b_s}^{(3,2)} & K_{b_s}^{(3,3)}
\end{bmatrix}_{3N_w \times 3N_w}
\]  \hspace{1cm} (15)

\[
K_{b_s}^{(1,1)} = k_w^1 \Psi_i^1(1)\Psi_j^1(1) \\
+ \frac{k_w^1}{L^1} \Psi_i^1(1)\Psi_j^1(1), \hspace{1cm} (16a)
\]

\[
K_{b_s}^{(1,2)} = -k_w^1 \Psi_i^1(1)\Psi_j^2(0) \\
- \frac{k_{w'}^1}{L^1} \Psi_i^1(1)\Psi_j^2(0), \hspace{1cm} (16b)
\]

\[
K_{b_s}^{(2,1)} = -k_w^1 \Psi_i^2(0)\Psi_j^1(1) \\
- \frac{k_{w'}^1}{L^1} \Psi_i^2(0)\Psi_j^1(1), \hspace{1cm} (16c)
\]

\[
K_{b_s}^{(2,2)} = k_w^1 \Psi_i^2(0)\Psi_j^2(0) \\
+ \frac{k_{w'}^1}{L^2} \Psi_i^2(0)\Psi_j^2(0) + \\
k_w^2 \Psi_i^2(1)\Psi_j^2(1) \\
+ \frac{k_{w'}^2}{L^2} \Psi_i^2(1)\Psi_j^2(1), \hspace{1cm} (16d)
\]

Similarly, the stiffness matrix of the highly stiff components in torsion is shown in Eq. 16.

\[
K_{b_t} = \begin{bmatrix}
K_{b_t}^{(1,1)} & K_{b_t}^{(1,2)} \\
K_{b_t}^{(2,1)} & K_{b_t}^{(2,2)}
\end{bmatrix}
\]  \hspace{1cm} (16e)

\[
K_{b_t}^{(1,1)} = -k_w^2 \Psi_i^2(1)\Psi_j^3(0) \\
- \frac{k_{w'}^2}{L^2} \Psi_i^2(1)\Psi_j^3(0), \hspace{1cm} (16f)
\]

\[
K_{b_t}^{(2,3)} = k_w^2 \Psi_i^3(0)\Psi_j^3(0) \\
+ \frac{k_{w'}^2}{L^3} \Psi_i^3(0)\Psi_j^3(0) \hspace{1cm} (16g)
\]

Finally, the mass matrix is

\[
M_b = \begin{bmatrix}
M_{b}(1,1) & 0 & 0 \\
0 & M_{b}(2,2) & 0 \\
0 & 0 & M_{b}(3,3)
\end{bmatrix}
\]  \hspace{1cm} (17)

where

\[
M_{b}^{(1,1)} = m^1 L^1 \int_0^l \Psi_i^1 \Psi_j^1 d\bar{y} \hspace{1cm} (18a)
\]

\[
M_{b}^{(2,2)} = m^2 L^2 \int_0^l \Psi_i^2 \Psi_j^2 d\bar{y} \hspace{1cm} (18b)
\]

\[
M_{b}^{(3,3)} = m^3 L^3 \int_0^l \Psi_i^3 \Psi_j^3 d\bar{y} \hspace{1cm} (18c)
\]

In the case of torsion, the stiffness and mass matrices are derived by substituting the following functions

\[
\bar{\theta}^{(1)} = \bar{y}^i, \hspace{1cm} i = 1, 2, ..., N_{\theta}^{(1)} \hspace{1cm} (19)
\]

\[
\bar{\theta}^{(c)} = \bar{y}^{i-1}, \hspace{1cm} i = 1, 2, ..., N_{\theta}^{(c)} \hspace{1cm} (20)
\]

into the series approximation of torsional deformation. The above functions are selected to satisfy the essential boundary conditions for each component. The first equation, Eq. 21, enforces cantilever beam boundary conditions, whereas the second equation, Eq. 22, enforces free-free boundary condi-
springs in torsion is:

\[ \ddot{\theta}^{(1)}(0) = 0, \quad \ddot{\theta}^{(1)}(1) \neq 0 \]  \hspace{1cm} (21)

\[ \ddot{\theta}^{(c)}(0) \neq 0, \quad \ddot{\theta}^{(c)}(1) \neq 0 \]  \hspace{1cm} (22)

The following torsional stiffness, springs and mass matrices result from substituting the above functions into Lagrange’s equation where \( i, j = 1, 2, \ldots, N_\theta \) are the number of terms included in the series.

\[
K_t = \begin{bmatrix}
K_t^{(1,1)} & 0 & 0 \\
0 & K_t^{(2,2)} & 0 \\
0 & 0 & K_t^{(3,3)}
\end{bmatrix}
\]  \hspace{1cm} (23)

\[
K_t^{(1,1)} = \frac{GJ(1)}{L} \int_0^1 \ddot{\theta}_i^{(1)} \ddot{\theta}_j^{(1)} \, d\bar{y} \]  \hspace{1cm} (24a)

\[
K_t^{(2,2)} = \frac{GJ(2)}{L} \int_0^1 \ddot{\theta}_i^{(2)} \ddot{\theta}_j^{(2)} \, d\bar{y} \]  \hspace{1cm} (24b)

\[
K_t^{(3,3)} = \frac{GJ(3)}{L} \int_0^1 \ddot{\theta}_i^{(3)} \ddot{\theta}_j^{(3)} \, d\bar{y} \]  \hspace{1cm} (24c)

The stiffness matrix of the highly stiff springs in torsion is:

\[
K_t^{(s)} = \begin{bmatrix}
K_t^{(1,1)}_{ts} & K_t^{(1,2)}_{ts} & 0 \\
K_t^{(2,1)}_{ts} & K_t^{(2,2)}_{ts} & K_t^{(2,3)}_{ts} \\
0 & K_t^{(3,2)}_{ts} & K_t^{(3,3)}_{ts}
\end{bmatrix}
\]  \hspace{1cm} (25)

where

\[
K_t^{(1,1)}_{ts} = k_\theta^{(1)} \ddot{\theta}_i^{(1)}(1) \ddot{\theta}_j^{(1)}(1), \]  \hspace{1cm} (26a)

\[
K_t^{(1,2)}_{ts} = -k_\theta^{(1)} \ddot{\theta}_i^{(1)}(0) \ddot{\theta}_j^{(2)}(0), \]  \hspace{1cm} (26b)

\[
K_t^{(2,1)}_{ts} = -k_\theta^{(1)} \ddot{\theta}_i^{(2)}(0) \ddot{\theta}_j^{(1)}(1), \]  \hspace{1cm} (26c)

\[
K_t^{(2,2)}_{ts} = k_\theta^{(2)} \ddot{\theta}_i^{(2)}(0) \ddot{\theta}_j^{(2)}(0) + k_\theta^{(2)} \ddot{\theta}_i^{(2)}(1) \ddot{\theta}_j^{(2)}(1), \]  \hspace{1cm} (26d)

\[
K_t^{(2,3)}_{ts} = -k_\theta^{(2)} \ddot{\theta}_i^{(2)}(0) \ddot{\theta}_j^{(3)}(0), \]  \hspace{1cm} (26e)

\[
K_t^{(3,2)}_{ts} = -k_\theta^{(2)} \ddot{\theta}_i^{(3)}(0) \ddot{\theta}_j^{(2)}(1), \]  \hspace{1cm} (26f)

\[
K_t^{(3,3)}_{ts} = k_\theta^{(2)} \ddot{\theta}_i^{(3)}(0) \ddot{\theta}_j^{(3)}(0). \]  \hspace{1cm} (26g)

Finally, the mass matrix is shown in Eq. 27 where \( r^{(c)} \) corresponds to the radius of gyration.

\[
M_t = \begin{bmatrix}
M_t^{(1,1)} & 0 & 0 \\
0 & M_t^{(2,2)} & 0 \\
0 & 0 & M_t^{(3,3)}
\end{bmatrix}
\]  \hspace{1cm} (27)

\[
M_t^{(1,1)} = m(1)L(1)r^{(1)} \int_0^1 \ddot{\theta}_i^{(1)} \ddot{\theta}_j^{(1)} \, d\bar{y} \]  \hspace{1cm} (28a)

\[
M_t^{(2,2)} = m(2)L(2)r^{(2)} \int_0^1 \ddot{\theta}_i^{(2)} \ddot{\theta}_j^{(2)} \, d\bar{y} \]  \hspace{1cm} (28b)

\[
M_t^{(3,3)} = m(3)L(3)r^{(3)} \int_0^1 \ddot{\theta}_i^{(3)} \ddot{\theta}_j^{(3)} \, d\bar{y} \]  \hspace{1cm} (28c)

For an unsymmetric section, inertial coupling of bending and torsion are accounted for with matrix \( B \) composed of components:

\[
B^{(i,j)} = d \int_0^1 \ddot{\theta}_i^{(i)} \Psi_j^{(j)} \, d\bar{y} \]  \hspace{1cm} (29)

The system of equations for both bending and torsion becomes:

\[
\begin{bmatrix}
M_b & B^T \\
B & M_t
\end{bmatrix}
\begin{bmatrix}
\ddot{\xi} \\
\xi
\end{bmatrix} = 0
\]  \hspace{1cm} (30)

The natural frequencies are computed by solving the eigenvalue problem resulting from substituting \( \xi = \exp(i\omega t) \) into previous equation.

Penalty Stiffnesses

As a check for the penalty approach, a cantilever beam [11] is analyzed. Two configurations of a multi-stepped beam are listed in Table 1. The first configuration is where the thickness is constant of all the three components \( t^{(c)} \). The beam depth \( d \), Modulus \( E \) and density \( \rho \) are held fixed. The nat-
ural frequencies are found equal to analytical values listed in [12, page 226]. The second configuration is where the thickness of second component $t^{(2)} = 3t^{(1)}$. The natural frequencies are found exactly equal to the one reported in this reference. The authors use a Lagrange multiplier approach to enforce continuity near the jump in thickness. However, their approach leads to a nontrivial characteristic equation that has to be solved for the natural frequencies. The penalty approach here has the advantage of retrieving the natural frequencies via a simple eigenvalue problem, which is more suitable for an automatic determination of the modes. The stiffness of the springs has to be chosen carefully to avoid inaccuracies and ill-conditioning of the system of equations. The values reported in Table 1 are for stiffness equal to $1 \times 10^6 \text{N/m}^2$ of the bending stiffness $EI$ and 10 terms of the series. The parameters for these results include a beam depth of 0.01 m and material properties for steel (Modulus of 200 GPa, density of 7800 kg/m$^3$). This procedure can easily be extended to include torsion modes.

Table 1: Dimensions and natural frequencies of multi-stepped cantilever beam.

<table>
<thead>
<tr>
<th></th>
<th>Config. 1</th>
<th>Config. 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L^{(1)}$ (m)</td>
<td>0.1</td>
<td>0.1</td>
</tr>
<tr>
<td>$L^{(2)}$ (m)</td>
<td>0.1</td>
<td>0.1</td>
</tr>
<tr>
<td>$L^{(3)}$ (m)</td>
<td>0.1</td>
<td>0.1</td>
</tr>
<tr>
<td>$t^{(1)}$ (m)</td>
<td>0.001</td>
<td>0.001</td>
</tr>
<tr>
<td>$t^{(2)}$ (m)</td>
<td>0.001</td>
<td>0.003</td>
</tr>
<tr>
<td>$t^{(3)}$ (m)</td>
<td>0.001</td>
<td>0.001</td>
</tr>
<tr>
<td>$\omega_1$ (rad/s)</td>
<td>57.1</td>
<td>52.5</td>
</tr>
<tr>
<td>$\omega_2$ (rad/s)</td>
<td>357.9</td>
<td>459.3</td>
</tr>
<tr>
<td>$\omega_3$ (rad/s)</td>
<td>1002.1</td>
<td>1005.5</td>
</tr>
<tr>
<td>$\omega_4$ (rad/s)</td>
<td>1963.7</td>
<td>3070.2</td>
</tr>
</tbody>
</table>

The bending penalty stiffness, $k^{(c)}_w$, is the maximum $EI^{(c)}/L^{(c)}$ multiplied by a large factor (i.e. $10^5, 10^6$). Similarly, the bending slope penalty stiffness, $k^{(c)}_w$, is based on the maximum $EI^{(c)}/L^{(c)}$, and the torsion penalty stiffness, $k^{(c)}_\theta$, is based on the maximum $GJ^{(c)}/L^{(c)}$. The penalty method is somewhat sensitive to the size of the multiplying factor depending on the number of sections under analysis. A cantilever beam with seven to ten sections is of interest for the purposes of modeling the inflatable wing with multiple telescoping spar sections, based on the packing conditions required. A study of the Config. 1 beam described in Table 1 with varying sections and penalty factors was conducted to determine the best penalty factor for a cantilever beam with seven to ten sections. The best penalty factor is $10^5$ based on the data shown in Fig. 1.
Conclusion

The Rayleigh-Ritz method with penalty approach has been fully described. The model is able to handle the discontinuity between rigid and inflatable sections. This method, unlike using Lagrange multipliers, allows for an automatic determination of natural modes; this is critical for the use of this method within the scope of a multidisciplinary design optimization. With careful selection of the penalty factor, a good approximation of the natural modes can be made. For the case of a beam with seven to ten sections, the best factor for the penalty terms was shown to be $10^5$. The penalty approach applied to the Rayleigh-Ritz method can easily be tailored to handle a wide range of designs without impact on the optimization process. This model was successfully incorporated in a multidisciplinary design optimization of a hybrid rigid and inflatable wing [13]. The development of this model and its successful application in optimization work will allow for further work in the analysis and design of ultra-large aspect-ratio wing structures.
References


