THE ACOUSTIC DIFFUSION MODEL FOR SINGLE AND COUPLED INTERIOR VOLUMES USING THE BOUNDARY ELEMENT METHOD
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Abstract
Recently, a new model for the propagation of sound in interior volumes known as the acoustic diffusion model has been explored as an alternative method for acoustic predictions and analysis. The model uses statistical methods standard in high frequency room acoustics to compute a spatial distribution of acoustic energy over time as a diffusion process. For volumes coupled through a structural partition, the energy consumed by structural vibration and acoustic energy transmitted between volumes is incorporated through an acoustic transmission coefficient. In this paper, Boundary Element Method (BEM) solutions to single and multiple volume models are developed. The integral form of the 3-D acoustic diffusion model is derived using the Laplace transform and Green’s Second Identity. The solution using the BEM is developed as well as an efficient Laplace transform inversion scheme to obtain both steady state and transient interior acoustic energy. Several volume configurations with varying geometry are examined as the diffusion model and its BEM solution are analyzed and compared to conventional room acoustics analysis methods. Advantages of this method over conventional methods such as computational efficiency, applicability to high frequencies, and robustness to different problems are revealed as the comparisons are made in different coupled volumes.

Introduction
The indoor propagation of sound, sometimes referred to as room acoustics or architectural acoustics, is an important branch of acoustics to study and understand. Humans spend much of their time indoors, whether at home, at work, or in school, so it comes as no surprise that noise is commonly experienced inside. Room acoustics has been studied for many years; traditional mathematical models include the wave equation, geometrical acoustics, and statistical analysis. However, new improvements are still being sought in its prediction today.

The most fundamental mathematical model of sound propagation is the acoustic wave equation, a partial differential equation which can be solved analytically for only very simple volume geometries. The wave equation in any more complex volume must be solved numerically using methods like the Finite Element Method (FEM), the Boundary Element Method (BEM), or the Finite Difference (FD) technique. These methods are computationally intensive and generally only applicable at low frequencies. Predicting sound indoors is also commonly carried out using techniques in geometrical acoustics, e.g. the image source method and ray tracing. Although accurate prediction of indoor sound is possible with these techniques, they rely on input acoustic properties of materials, e.g. absorption and scattering coefficients, which are inherently difficult to measure. Also, these methods rarely account for structural motion in the transmission of sound through partitions and, thus, encounter difficulties in problems of this type.

Statistical analyses of uniform sound energy in rooms are also common methods of predicting indoor acoustics. These techniques are only useful at high frequencies when there are no effects of individual resonant modes, if
the spatial distribution of sound is unimportant, and in only certain types of rooms. The types of rooms are restricted to relatively small rooms in which the reflective surfaces are not too distant from one another and no dimension of the room is so large that air absorption becomes important. For example, a long, flat room violates these restrictions and, thus, it requires alternative analysis tools.

Recent research has developed a new approach to statistically model room acoustics based on the diffusion theory of particles collisions in a gas. This new approach allows for the statistical analysis of the acoustic energy in a room as it varies in space. This research has shown that the acoustic diffusion model is an efficient, adaptive tool useful particularly for room acoustics. Picaut et al. compared one-dimensional (1-D) analytical solutions of the diffusion formulation to ray tracing solutions and experiments for very long rooms demonstrating good agreement. Valeau et al. derived different forms of the acoustic diffusion model for different types of absorption and demonstrated its flexibility to accommodate different types of sources, i.e. point, volume, and surface sources. They also compared FEM solutions of the diffusion model to statistical analysis, the image source method, and a ray tracing model for very long rooms demonstrating good agreement. Valeau et al.

In the previous research mentioned, solutions to the acoustic diffusion model have been limited to 1-D analytical solutions or to the use of the FEM for three-dimensional (3-D) solutions. This paper is focused on the development of a new, efficient solution to the acoustic diffusion model using the BEM for the indoor propagation of sound energy over time. It is expected that this tool could have wide-ranging applications in traditional room acoustics, transmission of sound into buildings, sound field predictions in space stations, and even sound fields inside of mechanical devices like mufflers.

First, an acoustic diffusion boundary integral equation for a single volume will be developed and solved with the BEM. Then, the BEM formulation of the diffusion model for multiple volumes will be presented using a simple transmission coefficient for structural coupling. Next, an initial validation of the BEM solution to the acoustic diffusion equation will be undertaken for a cubic room with uniform absorption through comparison with statistical analysis and ray tracing solutions. Then, comparisons of computational efficiency with the ray tracing method will be discussed for a room with simple geometry and surfaces with different, realistic absorption coefficients which vary with frequency. After that, diffusion BEM and ray tracing will be used to solve a simple transmission problem between two uniform cubic volumes. Finally, conclusions drawn from the work developed in this paper will be discussed.

**Theory**

**Single Volume**

The development of the boundary integral equation begins with the room acoustic diffusion equation,

\[
\frac{\partial \psi(r,t)}{\partial t} - D \nabla^2 \psi(r,t) = p(r,t),
\]

(1)
where $\psi(\mathbf{r},t)$ is the energy density at any point, $\mathbf{r}$, in the room at time, $t$; $D$ is the acoustic diffusion coefficient given by $D = \lambda c / 3$ with $\lambda = 4V_r / S_r$ being the mean free path length of the room of volume, $V_r$, and total surface area, $S_r$; $c$ is the speed of sound in air; $\nabla^2$ is the Laplacian operator; $\rho(\mathbf{r},t)$ is the acoustic source power per unit volume that the source occupies. A bold variable denotes a vector quantity.

The boundary condition at the bounding surfaces of the volume, $S$, is given by

$$D \frac{\partial \psi(\mathbf{r},t)}{\partial \mathbf{n}} + h(\mathbf{r})\psi(\mathbf{r},t) = 0, \quad \mathbf{r} \in S,$$  \hspace{1cm} (2)

where $h$ is the exchange coefficient given by $h(\mathbf{r}) = c \alpha(\mathbf{r})/4$ where $\alpha$ is the acoustic absorption coefficient of the surface, and $\mathbf{n}$ is the unit normal vector to the surface $S$ which is defined to be positive outward to $V$. The arbitrary initial acoustic energy density distribution at time zero is denoted with $\psi(\mathbf{r},0)$. A schematic showing the room volume features with the notation used is given in Fig. 1. Note that the schematic has been drawn with rectangular geometry for clarity, but the geometry at this point in the theoretical formulation is still arbitrary.

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The first volume integral accounts for the effect of the initial energy distribution in the domain. The second volume integral represents the acoustic energy density due to the arbitrary supplied acoustic power per volume of the source. Most often the bounds of this integral will reduce to a sub volume such as a point, a surface, or a smaller volume. The last integral represents the diffusion of the acoustic energy density accounting for the shape of the volume which is obtained by integrating only over the bounding surface of the domain.

The solution using the BEM involves discretization of the boundary into elements and numerically integrating across all of these small elements. This is performed for each node consecutively forming a linear system of equations for the energy at the nodes of the boundary. Once this system is solved, obtaining the energy at points in the volume is a simple post-processing of eq. (5).

To obtain physically meaningful results from the procedure discussed, the energy density in the Laplace domain, \( \Psi(\mathbf{r}, s) \), at any position of interest whether in the volume or on the surface must be inverted back to the time domain. Since \( \Psi(\mathbf{r}, s) \) is only known at discrete values of \( s = s_q \), an analytical inverse transform does not exist. Many numerical Laplace Transform inversion methods have been developed and studied\(^{13}\). Based on a comparison of these methods\(^{13}\), the procedure proposed by Liggett and Liu\(^{14}\) is used here to perform the inversion because of its simplicity. First, it is assumed that the time domain solution, \( \psi(\mathbf{r}, t) \), can be expanded in a series as

\[
\psi(\mathbf{r}, t) = \psi_{ss}(\mathbf{r}) + \sum_{q=1}^{Q} a_q \exp(-s_q t) \tag{7}
\]

where \( \psi_{ss}(\mathbf{r}) \) is the steady state solution obtained as \( t \to \infty \) and \( a_q \) are arbitrary constants to be determined. The Final Value Theorem is applied to eq. (5) to find the steady state form of the boundary integral equation needed in eq. (7), i.e. eq. (5) is multiplied by \( s \), the limit is taken as \( s \to 0 \), and then solved numerically. The coefficients, \( a_q \), are obtained by taking the Laplace Transform of eq. (7) as

\[
\Psi(\mathbf{r}, s) = \frac{\psi_{ss}(\mathbf{r})}{s} + \sum_{q=1}^{Q} \frac{a_q}{s + s_q} \tag{8}
\]

Equation (8) represents a set of \( Q \) algebraic equations which can be solved simultaneously for the values of \( a_q \). Then, eq. (7) is used to determine the acoustic energy density for any point in space as a function of time. It is recommended to use \( 5 \leq Q \leq 12 \) and that \( s_q \) are chosen as a geometric series\(^{14}\).

**Coupled Volumes**

For simplicity in the equations, the multiple volume formulation will be developed for two coupled acoustic volumes as shown in Fig. 2. However, this formulation can be easily extended used for any number or configuration of volumes and coupling surfaces. The domain of the first volume will be referred to as \( V_1 \) and the domain of the second as \( V_2 \). The domains of the bounding surfaces of each volume are split into two sub-surfaces each. The domain of the boundary exclusive to Volume 1 is \( S_1 \) and exclusive to Volume 2 is \( S_2 \). The domain of the boundary of Volume 1 on the coupling surface is \( S_{c1} \) and the boundary of Volume 2 on the coupling surface is \( S_{c2} \). The unit normal vector pointing outward from \( S_1 \) or \( S_{c1} \) is \( \mathbf{n}_1 \) and the unit normal pointing outward from \( S_2 \) or \( S_{c2} \) is \( \mathbf{n}_2 \). The position vector can be located in any of these defined domains. From here on, the subscript \( v \) will denote a quantity belonging to the current volume and \( u \) will denote a quantity belonging to the adjacent volume.

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The diffusion equation for the two volumes is given by

\[
\frac{\partial \psi_v(r,t)}{\partial t} - D_v \nabla^2 \psi_v(r,t) = f_v(r,t), \quad r \in V_v
\]  

(9)

where \( \psi_v \) represents the energy density in the \( v^{th} \) volume or on its bounding surface, \( D_v \) the diffusion coefficient in the \( v^{th} \) volume, and \( f_v \) the supplied acoustic power per unit volume in the \( v^{th} \) volume. The boundary conditions for the four surfaces are given by

\[
\begin{align*}
D_v \frac{\partial \psi_v(r,t)}{\partial n_v} + h_v(r) \psi_v(r,t) &= 0, & r \in S_v \\
D_v \frac{\partial \psi_v(r,t)}{\partial n_v} + h_v(r) \psi_v(r,t) &= \frac{c r_a(r)}{4} \psi_a(r,t), & r \in S_{cv}
\end{align*}
\]  

(10)

where \( h_v(r) = c \left[ \alpha_v(r) + \tau_v(r) \right] / 4 \), \( \alpha_v(r) \) is the absorption coefficient of the bounding surface of the \( v^{th} \) volume and \( \tau_v(r) \) is the transmission coefficient of the coupling partition associated with energy flowing out of the \( v^{th} \) volume. After conversion to the Laplace domain and application of Green’s Second Identity, the integral equation governing either volume is given by

\[
\beta \psi_v(r',s) = \frac{1}{D_v} \int G_v(r | r',s) \psi_v(r,0) dV +
\]

\[
\frac{1}{D_v} \int G_v(r | r',s) F_v(r,s) dV -
\]

\[
\int_{S_v + S_{cv}} \frac{h_v(r)}{D_v} \psi_v(r,s) G_v(r | r',s) dS +
\]

\[
\int_{S_v + S_{cv}} \psi_v(r,s) \frac{\partial G_v(r | r',s)}{\partial n_v} dS +
\]

\[
\frac{c}{4D_v} \int r_a(r) \psi_a(r,s) G_v(r | r',s) dS.
\]  

(11)

It should be noted that the integral equations in each volume are the same as in the single volume case with the addition of one more integral of the energy density transmitted from the coupled volume over the coupling surface. Again, this equation can be solved with the BEM by discretizing each boundary into elements and numerically integrating across each consecutively for each node. It is important to note that the common surfaces have two nodes at each position, one associated with the energy on either side of the partition.

**Numerical Studies**

**Uniform Cubic Room**

To provide an initial validation of the BEM diffusion tool developed, the test case consists of a cubic room with uniform absorption. This example problem was selected because it can be compared directly to a published FEM solution of the acoustic diffusion equation. In addition, the cubic room with uniform absorption most accurately matches the assumption in the Sabine statistical theory for room acoustics.

The room has dimensions of 10 x 10 x 10 m and the absorption coefficients of the room surfaces were set to a uniform value of 0.1. In this example, a monopole source radiating with a step input at zero time and a sound power level of 100 dB is placed at the center of the room. Initially, there is no sound energy...
present in the room. A schematic of the problem is shown in Fig. 3.

Fig. 3. Geometry of the uniform cube example used for validation of the BEM solution to the acoustic diffusion model.

The BE mesh has 6672 elements providing 10.5 elements per mean free path length. A 3-D grid of receivers is placed in the room with a spacing of 1 m in the three rectangular directions parallel to the room surfaces. Receivers on the surfaces of the room were not considered due to the inability of the ray tracing code used to handle such receivers. In addition, a receiver was not placed at the source location because of singularities in the Green’s function and analytical formulas when the distance between the source and receiver is zero. Thus, 728 receivers were used in this example which will provide detailed results throughout the room as well as average results in the room. The time domain solution was calculated to two seconds. It was decided that the Laplace variable from $10^{-3}$ to $10^3$ is an appropriate range containing enough initial and steady state information. Ten values of the Laplace variable taken as a geometric series over this range are sufficient to describe the energy in the room as a function of time.

The BEM results are compared to results using the Sabine method, the FEM, and a commercial ray tracing tool. The Sabine model is a statistical model that computes the average energy density in the room and that the absorption of the room can be described by a single average room absorption coefficient\(^3\). The uniform energy density as a function of time can be found analytically for this example using the Sabine model. For comparison with this analytical model, the energy density time history obtained from the BEM was averaged across all receivers in the room. The ray tracing technique has proven to provide the most accurate results compared to experiments among currently available room acoustics prediction tools\(^5,10\). For this reason, the total energy at each receiver was computed using the ray tracing code, Ramsete.

The source radiated about 16000 rays and there was no scattering of the rays’ energy from the room walls. The total energy at each time step was calculated by summing the energy of each ray passing through the receiver at all previous time steps. The total energy computed using ray tracing was also averaged for all receivers in the room. As mentioned, the same example problem was solved for the steady state response using a FEM formulation\(^6\). The steady state FEM solution was calculated with 5000 unstructured, linear, Lagrange-type elements\(^6\).

Both the steady state and the transient average energy density responses using the BEM method developed here are shown in Fig. 4. Also plotted in this figure are the transient energy density computed analytically (Sabine method) and using ray tracing (Ramsete) as well as the steady state response at the average receiver distance from the source computed with the FEM solution\(^6\).

Fig. 4. Average energy density time histories calculated with different methods.

The plot demonstrates very good agreement with differences in steady state
energy density levels obtained with the BEM of less than 1 dB with the statistical theory, about 0.3 dB with the FEM, and less than 0.1 dB with ray tracing. It is interesting to note that the BEM solution to the diffusion equation demonstrates the most conservative prediction with the lowest energy density between all methods. The initial exponential increasing trend demonstrates no significant difference between methods.

Assuming uniform absorption of the room, the Sabine model can predict the steady state SPL analytically at any point in the room. The steady state SPLs of the receivers calculated with the BEM and with the ray tracing code are plotted according to the receiver distance from the source from 1 to 7 m in Fig. 5 and compared to the statistical Sabine model and the steady state SPL calculated with the FEM as reported graphically. Because the diffusion equation only models the reverberant energy in the room, the BEM solution was corrected by adding the steady state contribution from the direct field radiated by the source. While the steady state SPL is actually calculated using the BEM, the steady state level computed with the ray tracing method is simply taken to be the latest value of the total energy, i.e. the total energy received after two seconds have passed.

**Fig. 5.** Steady state SPL at receivers plotted against the distance from the source calculated with different methods.

The BEM results predict the highest levels close to the source, but differ from all other methods by less than 1 dB. In fact, the BEM and FEM solutions to the diffusion equation agree with the statistical theory to within 1 dB over the whole range plotted crossing from over to under prediction compared to Sabine theory about 3 m away from the source. However, the rate of decay of sound energy with distance from the source is the greatest in the corrected BEM results. While the ray tracing results match closest to the statistical theory close to the source, they tend to disagree by greater amounts past about 2.5 m distance from the source. Further from the source, the BEM results tend to agree best with the ray tracing results. Also, the ray tracing results demonstrate significant ambiguity between receivers which are the same distance from the source showing up as vertical lines in the plot and do not demonstrate a smooth decaying trend as the source distance increases. These ambiguities should not be present as the problem is perfectly symmetric and can be attributed to numerical errors. The BEM solution demonstrates similar ambiguities at larger distances from the source but to a much lesser degree and can be attributed to round-off error. Overall, excellent agreement is demonstrated between the different approaches.

**Computational Efficiency**

In this section, a more complex room will be studied using the acoustic diffusion equation and its BEM solution. A detailed comparison of the computational efficiency of the diffusion equation BEM solution to a ray tracing software will be given. Results demonstrated good agreement between the diffusion BEM and ray tracing but will not be presented in this paper, as validation is provided in other studies and the computational efficiency presented more interesting conclusions.

The room under investigation in this section is a rectangular room with approximate dimensions 4.56 x 2.82 x 2.70 m...
as shown in Fig. 6. The absorption coefficients of the room surfaces were set to more realistic values dependent on frequency. The absorption coefficients of the room surfaces were set to more realistic values dependent on frequency.3.

In this case, a monopole source radiating with a step input at zero time and a sound power level of 96 dB for all octave bands is placed in a corner of the room with coordinates of (0.5, 0.5, 0.16) m with respect to the closest corner. Initially, there is no sound energy present in the room. Two 3-D grids of receivers of different sizes are placed in the room; the first is a 9 x 9 x 9 grid omitting receivers at the boundaries and the second is a 16 x 16 x 16 grid also omitting receivers at the boundaries. Thus, 729 receivers make up the first grid while the second consists of 4096 receivers. Both the transient BEM and ray tracing solutions were calculated to one and two second durations for the small receiver grid. The solution was calculated to one second for the large receiver grid.

The boundary mesh for the BEM contains 7680 triangular elements and 3842 nodes giving about 10.7 elements per mean free path length, enough for convergence. The time domain solution using the BEM was calculated using six values of the Laplace variable over the range 10^{-2} to 10^{4}. To obtain the solution for all octave bands considered, the BEM solution was simply calculated once for each octave band, adjusting the absorption coefficients to the appropriate values, for a total of ten bands. The ray tracing solution was calculated with 16384 rays, the number required to obtain a deterministic or non-approximate solution up to 1.25 s long; there was no diffraction or surface diffusion included.

The total computation times for each solution method was calculated excluding the time taken to write output files using a laptop personal computer with a dual core processor clocked at 2.10 GHz and 4 GB of RAM. The diffusion BEM tool was written in the Fortran programming language. Care was taken to ensure similar memory and processor states for each method through complete shutdown and boot up of the computer before each run. The computation times required to calculate the steady state BEM solution, the transient BEM solution, and the ray tracing solution are tabulated for both small and large receiver grids in Table 1.

Table 1: Total computation times in seconds for steady state BEM, transient BEM, and ray tracing solutions for different receiver grids over ten octave bands.

<table>
<thead>
<tr>
<th>Receiver Grid (#)</th>
<th>Computation Time (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Steady BEM</td>
</tr>
<tr>
<td>Small (729)</td>
<td>1</td>
</tr>
<tr>
<td>Large (4096)</td>
<td>1</td>
</tr>
</tbody>
</table>

For the smaller receiver grid, the time taken to compute the steady state BEM solution was about one third of that taken by the ray tracing for one second of response. When the response to two seconds was calculated, the steady BEM solution took only about one eighth of the computation time than the ray tracing solution. The transient BEM solution took over three times as long as the ray tracing procedure when calculated to one second, but only about ten percent longer to two seconds. When more receivers are desired, the ray tracing computation time increases faster than that of the BEM solution. For the large receiver grid, the steady state BEM solution took about one tenth of the time taken to compute the ray tracing solution and
the full time dependent BEM solution took about 5% less time than ray tracing.

When the number of receivers is increased about 5.6 times, the computation time of either BEM solution increases by only about 1.7 times. However, the computation time of the ray tracing solution increases about 5.7 times, directly proportional to the number of receivers. The small increase in the BEM computation time is due to the fact that the size of the mesh is not increasing; the time taken to calculate and invert the matrix of coefficients to compute the boundary solution is the same. The number of post processing steps is simply increased by the number of receivers, which proves to be a great advantage of the BEM over ray tracing when the number of desired receivers is large.

When the desired response time is increased two times for the smaller receiver grid, the BEM solutions’ computation times increase by a matter of seconds while the ray tracing computation time increases almost three fold. This does not take into account the increase in number of rays required to compute a convergent ray tracing solution; the ray tracing solution is approximate past 1.25 s in the two second long simulation with only 16384 rays.

Exactly when one method is more efficient than the other is dependent on the problem, but there will always exist a point when the BEM is more efficient than ray tracing.

Coupled Uniform Cubic Rooms

To examine the BEM solution of the acoustic diffusion equation for volumes coupled by a partition, a simple room configuration previously studied was simulated. The geometry, shown in Fig. 7, consists of two, cubic, 5 x 5 x 5 m volumes with a 25 m² coupling partition. A monopole source is placed in the corner of one of the rooms 2 m away from each surface which radiates a sound power level of 100 dB. Three-dimensional grids of receivers are placed in each room with a uniform spacing of 0.5 m, skipping the source position and not placing receivers on the boundaries. The boundaries of each volume, excluding the coupling surface, have a uniform acoustic absorption coefficient of 0.1 and the transmission loss (TL) of the coupling boundary is varied from 0 to 50 dB and the steady state energy is calculated at each receiver using the BEM.

Fig. 7. Example problem schematic used for initial validation of the BEM solution to the coupled acoustic diffusion equation showing geometry and source location.

The steady state SPL in each volume was then calculated using a variety of methods for different values of the TL of the coupling partition. The statistical formulation presented previously was used as a baseline analytical comparison since the simple configuration attempts to accurately represent the assumptions made in statistical theory. The diffusion BEM solution was computed using a mesh refined to give about 19 elements per mean free path length in each volume. A ray tracing solution was also calculated to three seconds using about 66000 rays. All receivers in the same volume were averaged to determine an average energy level in each volume when using the diffusion BEM and ray tracing. The average SPL in each volume calculated using the three different methods is plotted against the coupling partition TL in Fig. 8.
Fig. 8. Average SPL in each volume calculated using statistical theory, the diffusion BEM, and ray tracing for different partition TL’s.

The average steady state SPL calculated using diffusion BEM agrees with statistical theory to within 2 dB in each volume for all partition TL’s; these two methods agree to within 1 dB for all TL’s in Volume 1. In Volume 1, all three methods agree to within 2 dB for all partition TL’s. However, the ray tracing results differ from statistical theory by greater than 10 dB for higher values of TL (i.e. 50 dB) and more than 3 dB at the lowest TL (i.e. 0.1 dB). The difference between ray tracing and the diffusion BEM in Volume 2 is not as pronounced, but there are still large disparities, e.g. about a 10 dB difference when the TL is greater than 20 dB. Obviously, ray tracing encounters difficulty as rays pass through a structural partition. However, all three methods capture the same linear decrease in SPL with increasing TL.

Conclusions

Single and multiple volume acoustic diffusion models have been investigated using a BEM solution. This new solution method demonstrates accuracy in several different volume configurations when compared to other acoustic diffusion model solutions and more conventional techniques in room acoustics, i.e. statistical theory and ray tracing. In interior acoustic problems including structural transmission, the diffusion BEM demonstrates a greater ability to accurately describe the phenomenon than does ray tracing. The BEM solution also demonstrates considerable efficiency advantages in comparison to ray tracing, thus proving a valuable tool in the study of interior acoustics.

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References